Weyl Functional Calculus and the Quantum Harmonic Oscillator

Christoph Kriegler (Clermont-Ferrand, France), in collaboration with Cédric Arhancet (Albi, France), Lukas Hagedorn (Kiel, Germany) and Pierre Portal (Canberra, Australia)

ICMS 2025 Edinburgh

Mihlin's (1956) and Hörmander's (1960) theorem on Fourier multipliers on $L^p(\mathbb{R})$

Let $m : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ be a differentiable function such that $\sup_{t\in\mathbb{R}^*} |m(t)|$, $\sup_{t\in\mathbb{R}^*} |tm'(t)| < \infty$. Let $1 < p < \infty$. Then the operator $N_m : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ given by

$$N_m(f)(x) = \mathcal{F}^{-1}[m\hat{f}](x) = \frac{1}{2\pi}\int_{\mathbb{R}} m(t)\hat{f}(t)e^{ixt}dt$$

is bounded.
We denote $M^1(\mathbb{R})$ the class of such functions m with $\|m\|_{M^1(\mathbb{R})} = \max\{\sup_{t\in\mathbb{R}^*} |m(t)|, \sup_{t\in\mathbb{R}^*} |tm'(t)|\}$.
There is also a d-dimensional Mihlin theorem, and in its form for radial functions $m : \mathbb{R}^d \to \mathbb{C}$, it reads as
$\|N_m\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim \|m\|_{M^k(\mathbb{R}_+)}$ with $k > \frac{d}{2}$ and
$\|m\|_{M^k(\mathbb{R}_+)} = \max\left\{\sup_{t>0} |m(t)|, \ldots, \sup_{t>0} |t^k m^{(k)}(t)|\right\} < \infty.$

Interpretation through the C_0-group of translations

One can rewrite

$$N_m f(x) = \check{m} * f(x) = \int_{\mathbb{R}} \check{m}(s) f(x-s) ds = \int_{\mathbb{R}} \check{m}(s) T_s f(x) ds$$

with $T_s f(x) = f(x-s)$ the translation. $T_s : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is uniformly bounded, and moreover $(T_s)_{s\in\mathbb{R}}$ is a C_0-group.

Definition: C_0-group

Let X be a Banach space and for $s \in \mathbb{R}$, $T_s : X \to X$. Then $(T_s)_{s\in\mathbb{R}}$ is called C_0-group if

1. $T_0 = \mathrm{Id}_X$,
2. $T_{s+t} = T_s \circ T_t$, $(s, t \in \mathbb{R})$,
3. $T_s(x) \to x$ for $s \to 0$, $(x \in X)$.

Transference principle

Question

For which C_0-groups $(T_s)_{s\in\mathbb{R}}$, on which spaces X and for which functions $m : \mathbb{R} \to \mathbb{C}$, is the operation

$$M_m(x) = \int_{\mathbb{R}} \check{m}(s) T_s(x) ds$$

bounded $X \to X$?

The question is formulated very generally. The following is a first classical result.

Theorem (Transference principle, Coifman-Weiss 1977)

Let $(T_s)_s$ be a C_0-group on X such that $C := \sup_{s\in\mathbb{R}} \|T_s\|_{X\to X} < \infty$. Let $(S_s)_s$ be the C_0-group of translations on $L^p(\mathbb{R}, X)$. Let $m : \mathbb{R} \to \mathbb{C}$, M_m be the operator associated to $(T_s)_s$ and N_m the operator associated with $(S_s)_s$. Then

$$\|M_m\|_{X\to X} \le C^2 \|N_m\|_{L^p(\mathbb{R},X)\to L^p(\mathbb{R},X)}.$$

Mihlin's theorem on $L^p(\mathbb{R}, UMD)$

Question:

For which Banach spaces X do we have an estimate of the norm $\|N_m \otimes \mathrm{Id}_X\|_{L^p(\mathbb{R},X)\to L^p(\mathbb{R},X)}$?

- if $X = L^p(\Omega)$, then by Fubini,

$$\begin{aligned}
&\left\|N_m \otimes \mathrm{Id}_{L^p(\Omega)}\right\|_{L^p(\mathbb{R},L^p(\Omega))\to L^p(\mathbb{R},L^p(\Omega))} \\
&= \left\|\mathrm{Id}_{L^p(\Omega)} \otimes N_m\right\|_{L^p(\Omega,L^p(\mathbb{R}))\to L^p(\Omega,L^p(\mathbb{R})} \\
&= \|N_m\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} .
\end{aligned}$$

  So with Mihlin's theorem,
  $\|N_m \otimes \mathrm{Id}_X\|_{L^p(\mathbb{R},X)\to L^p(\mathbb{R},X)} \leq C_p \|m\|_{M^1(\mathbb{R})}$.
- Theorem (Bourgain, McConnell, Zimmermann): If X is a UMD space, then $\|N_m \otimes \mathrm{Id}_X\|_{L^p(\mathbb{R},X)\to L^p(\mathbb{R},X)} \leq C_{p,X} \|m\|_{M^1(\mathbb{R})}$. As an approximate rule, any reasonable reflexive space is UMD: For $1 < p < \infty$: ℓ^p, $L^p(\Omega)$, $W^{m,p}(\mathbb{R})$, S^p, ...

Functional calculus of the generator of a C_0-group

Notation:

For a C_0-group $(T_s)_{s\in\mathbb{R}}$, we write $iAx = \lim_{s\to 0}\frac{1}{s}(T_s(x)-x)$ the generator. Then

$$\frac{1}{2\pi}M_m x = \frac{1}{2\pi}\int_{\mathbb{R}} \check{m}(s)e^{isA}x ds = m(A)x$$

becomes the functional calculus of the generator.

For example, if $T_s f(x) = f(x-s)$ on $L^p(\mathbb{R})$, then $A = i\frac{d}{dx}$.
By putting the above results together, one obtains

Theorem (Coifman-Weiss (1977), Bourgain (1983), McConnell (1984), Zimmermann (1989), Hieber-Prüss (1998))

Let $(T_s)_{s\in\mathbb{R}}$ be a bounded C_0-group acting on a UMD Banach space X. Let iA be its generator. Then

$$\|m(A)\|_{X\to X} \leq C\,\|m\|_{M^1(\mathbb{R})}.$$

Families of C_0-groups

From now on, we consider finite families of bounded C_0-groups which act on a same Banach space.

Example

Let $d \in \mathbb{N}$, $1 \leq p < \infty$, and $X = L^p(\mathbb{R}^d)$.
For $k = 1, \ldots, d$, let $e^{itA_k}(f)(x) = f(x - te_k)$ be the translation in direction k.
Then $(e^{itA_1})_{t\in\mathbb{R}}, (e^{itA_2})_{t\in\mathbb{R}}, \ldots, (e^{itA_d})_{t\in\mathbb{R}}$ form a family of d bounded C_0-groups that commute, on $L^p(\mathbb{R}^d)$, with generators $iA_k = -\partial_k$.

Multiplication C_0-groups

There is another natural family of bounded C_0-groups on $L^p(\mathbb{R}^d)$.

Example

Let $d \in \mathbb{N}$, $1 \leq p < \infty$, and $X = L^p(\mathbb{R}^d)$.
For $k = 1, \ldots, d$, let $e^{itB_k} f(x) = e^{itx_k} f(x)$ be multiplication operators.
Then $(e^{itB_1})_{t\in\mathbb{R}}, (e^{itB_2})_{t\in\mathbb{R}}, \ldots, (e^{itB_d})_{t\in\mathbb{R}}$ form a family of d bounded commuting C_0-groups, with generators $iB_k = ix_k$.

Families of non-commuting C_0-groups

Theorem (Lancien-Lancien-Le Merdy (1998), Kalton-Weis (2001), Hytönen (2004), Dore-Venni (2005))

Let $C_1, \ldots, C_d$ be generators of d bounded and commuting C_0-groups acting on the same UMD space. Then

$$\|m(C_1, \ldots, C_d)\| \leq C \sup\left\{ |t|^{|\alpha|} |D^\alpha m(t)| : \ \alpha_k \in \{0,1\}, \ |\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1 \right\}$$

Question

Is this Theorem applicable to $\mathcal{A} = (A_1, \ldots, A_d, B_1, \ldots, B_d)$ with $A_k = i\partial_k$ and $B_k = x_k$? Do the C_0-groups e^{itA_k} and e^{itB_j} commute?

Answer

No, but almost yes! All groups commute except for the rule

$$e^{itA_k} e^{isB_k} = e^{-ist} e^{isB_k} e^{itA_k}.$$

Definition of Θ-Weyl tuples

Inspired by the commutation relations of the preceding slide, we define the following generalisation of translation and multiplication groups.

Definition Θ-Weyl tuple

Let $iX_1, \ldots, iX_n$ be generators of bounded C_0-groups that act on a same Banach space X. Let $\Theta \in \mathbb{R}^{n \times n}$ be an anti-symmetric matrix ($\Theta^T = -\Theta$). We call $(X_1, \ldots, X_n)$ a Θ-Weyl tuple if

$$e^{itX_k} e^{isX_j} = e^{i\Theta_{kj}ts} e^{isX_j} e^{itX_k} \quad (k, j = 1, \ldots, n,\ t, s \in \mathbb{R}).$$

So $(i\partial_1, \ldots, i\partial_d, x_1, \ldots, x_d)$ is a $\begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}$-Weyl tuple on $L^p(\mathbb{R}^d)$.

The Θ-Weyl functional calculus

Definition of the Θ-Weyl functional calculus

Let $\mathcal{X} = (X_1, \ldots, X_n)$ be a Θ-Weyl tuple. For $t \in \mathbb{R}^n$, we put

$$e^{it\cdot\mathcal{X}} = e^{-\frac{1}{2}i\sum_{k>j}\Theta_{kj}t_k t_j}\prod_{k=1}^{n} e^{it_k X_k}.$$

Then we have the relation

$$e^{it\cdot\mathcal{X}}e^{is\cdot\mathcal{X}} = e^{\frac{1}{2}i\langle t,\Theta s\rangle}e^{i(t+s)\cdot\mathcal{X}}.$$

Next we define for $m \in S(\mathbb{R}^n)$,

$$m(\mathcal{X}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{m}(t)e^{it\cdot\mathcal{X}}dt,$$

the Θ-Weyl functional calculus.

Functional calculus of Θ-Weyl tuples

Question

Does the Θ-Weyl functional calculus admit Mihlin M^k bounds as it was the case for $n = 1$ (a single C_0-group)/ for the case of $\Theta = 0$?

Let us examinate the two steps of the proof for the case $n = 1$.

Θ-transference principle

1st step: Transference principle? One has the following result:

Theorem Transference Principle (van Neerven-Portal (2020), Arhancet-Hagedorn-K.-Portal (2023))

Let $\Theta \in \mathbb{R}^{n\times n}$ be anti-symmetric. There exists a Θ-Weyl tuple $A_{univ} = (A_1, \ldots, A_n)$ on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ with the following property. For $\mathcal{X}$ any Θ-Weyl tuple which acts on any Banach space X, and $m \in S(\mathbb{R}^n)$,

$$\|m(\mathcal{X})\|_{B(X)} \lesssim \|m(A_{univ} \otimes \mathrm{Id}_X)\|_{B(L^p(\mathbb{R}^n,X))}.$$

The tuple A_{univ} is explicit, the A_k are linear combinations of the ∂_k and the x_j.

Functional calculus of the universal Θ-Weyl tuple

2nd step: Functional calculus of the universal tuple $A_{univ} \otimes \mathrm{Id}_X$ on $L^p(\mathbb{R}^n, X)$.
Problem: In contrast with the translation-multiplication tuple $(i\partial_1, \ldots, i\partial_d, x_1, \ldots, x_d)$, the functional calculus of the Θ-Weyl tuple A_{univ} does not consist of pseudo-differential operators.
Consequently, one does not know reasonable bounds of the functional calculus of A_{univ}.
In order to obtain a result: In the following, we restrict the functional calculus of A_{univ} to radial spectral multipliers.
That is to say, we put $B = \sum_{k=1}^n A_k^2$, where $A_{univ} = (A_1, \ldots, A_n)$ and consider later $m : (0, \infty) \to \mathbb{C}$ which will yield a bounded operator $m(B)$.

Functional calculus of the universal Θ-Weyl tuple 2

We have the following result:

Lemma

$B = \sum_{k=1}^{n} A_k^2$ is the generator of a contractive C_0-semigroup $(T_t)_{t \geq 0}$ on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.
We have $\sigma(B) \subseteq [0, \infty)$ (even in $[\alpha, \infty)$ for an $\alpha = \alpha(\Theta) > 0$, $\Theta \neq 0$).
$(T_z)_z$ is analytic for $\mathrm{Re}\, z > 0$.
Moreover, for all z with $\mathrm{Re}\, z > 0$, T_z has a rapidly decaying integral kernel.

Universal Weyl tuple on Schatten classes

Definition Schatten class

Let $1 < p < \infty$. Then
$S^p = \{T : \ell^2 \to \ell^2 : \; T \text{ compact and } \mathrm{tr}(|T|^p) < \infty\}$ is a UMD Banach space with norm $\|T\|_{S^p} = \mathrm{tr}(|T|^p)^{\frac{1}{p}}$.

Theorem (Arhancet-Hagedorn-K.-Portal (2023))

Let Θ be anti-symmetric and A_{univ} the universal Θ-Weyl tuple on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Let B be the sum of squares which generates the C_0-semigroup associated with A_{univ}. Then
$\|m(B \otimes \mathrm{Id}_{S^p})\|_{B(L^p(\mathbb{R}^n, S^p))} \lesssim \|m\|_{M^k(\mathbb{R}_+)}$ with $k \geq \frac{n+1}{2}$.
The proof is based on a new method to obtain square function estimates in the space $L^p(\mathbb{R}^n, S^p)$ and [González-Pérez–Junge–Parcet 2017].

Application: The Moyal-Groenewold plane 1

Let $\theta \in \mathbb{R}^{d\times d}$ anti-symmetric and $s \in \mathbb{R}^d$.
One defines unitaries $\lambda_{\theta,s} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$\lambda_{\theta,s}(\xi)(t) = e^{\frac{1}{2}i\langle s,\theta t\rangle}\xi(t-s)$$

(twisted translation).
$\theta = 0$: $\lambda_{\theta,s} \cong e^{is(\cdot)} \in L^\infty(\mathbb{R}^d)$.
There is a non-commutative L^p space $L^p(\mathbb{R}^d_\theta, \tau)$ generated by these unitaries, whose typical elements are of the form

$$\lambda_\theta(f) = \int_{\mathbb{R}^d} f(s)\lambda_{\theta,s} ds$$

for some f.
Then on this non-commutative L^p space, there are operators

- $e^{tD_k}\lambda_{\theta,s} = e^{its_k}\lambda_{\theta,s}$ (translation, or derivation),
- $e^{itX_k}\lambda_\theta(f) = \lambda_{\theta,te_k}\lambda_\theta(f)$ (multiplication).

The tuple $(D_1, \ldots, D_d, X_1, \ldots, X_d)$ is a Θ-Weyl tuple with $\Theta = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & \theta \end{pmatrix} \in \mathbb{R}^{2d\times 2d}$.

Application: The Moyal-Groenewold plane 2

Theorem (Arhancet-Hagedorn-K.-Portal (2023))

Let $1 < p < \infty$. Let $(iD_1, \ldots, iD_d, X_1, \ldots, X_d)$ be the Θ-Weyl tuple as above on the non-commuative L^p space $L^p(\mathbb{R}^d_\theta, \tau)$. Let $B = \sum_{k=1}^d -D_k^2 + X_k^2$ be the sum of squares operator: the quantum harmonic oscillator.
(If $\theta = 0$, then $B = \sum_{k=1}^d -\partial_k^2 + x_k^2 = -\Delta + |x|^2$).
Then

$$\|m(B)\|_{B(L^p(\mathbb{R}^d_\theta,\tau))} \lesssim \|m\|_{M^k(\mathbb{R}_+)}$$

with $k > d + \frac{1}{2}$.

Thank you very much for your attention