Codifferential Calculi and Quantum Homogeneous Spaces

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New perspectives in quantum representation theory
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Noncommutative (Differential) Geometry

Differential Geometry | Noncommutative Geometry

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Differential Geometry			Noncommutative Geometry
		$C^\infty(M)$ smooth functions	
	$\rightsquigarrow$ (up)		
M manifold	$\rightsquigarrow$	$\Omega^1(M)$ 1-forms	
	$\rightsquigarrow$ (down)		
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Philosophy

A noncommutative manifold is a noncommutative algebra A together with a first order differential calculus on A.

Differential Calculi

- Fix a field $\mathbb{k}$ with $\mathrm{char}(\mathbb{k}) \neq 2$.

Definition

A first order differential caclulus (fodc) on a $\mathbb{k}$-algebra A is an A-bimodule Ω^1 together with a derivation $d\colon A \to \Omega^1$ such that $A \otimes A \to \Omega^1, a \mapsto ad(b)$ is surjective.

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- Problem: The universal fodc tends to be the wrong choice for a fodc; for instance, $H^\bullet(\Omega^\bullet_{u,\mathrm{max}}) = \mathbb{k}$. Thus, arbitrary associative $\mathbb{k}$-algebras are too general.

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- Let $q \in \mathbb{C}^\times \setminus \sqrt[\mathbb{N}]{1}$. Then we have the following q-analog:

$$\mathcal{O}_q(L_S) \twoheadleftarrow \mathcal{O}_q(G) \hookleftarrow \mathcal{O}_q(G/L_S) := \mathcal{O}_q(G)^{\mathrm{co}\ \mathcal{O}_q(L_S)}$$

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- Note: for $q \in (0, \infty) \setminus \{1\}$ all the algebras above admit $*$-structures and in the limit $q \to 1$, $\mathcal{O}_q(G/L_S)$ recovers the real coordinate ring of G/P_S.

The Heckenberger–Kolb Differential Calculi

Definition

If the subset of simple roots S is given by removing a cominuscule root, we call $\mathcal{O}_q(G/L_S)$ irreducible quantized flag mainfold.

Theorem (Heckenberger–Kolb '06)

Let $B = \mathcal{O}_q(G/L_S)$ be an irreducible quantized flag manifold. Then there exist first order differential calculi $\Omega_q^{(1,0)}(G/L_S)$ and $\Omega_q^{(0,1)}(G/L_S)$ on B such that the maximal prolongations satisfy

$$\dim \Omega_q^{(r,0)}(G/L_S) = \dim \Omega_q^{(0,r)}(G/L_S) = \binom{m}{r}, \quad m = \dim_{\mathbb{C}} G/P_S.$$

The Quantized Enveloping Algebra Side

- Quantized flag manifolds have the following incarnation on the Lie algebraic side
$$U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g}) \twoheadrightarrow C_q(\mathfrak{g}/\mathfrak{l}_S) := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} \mathbb{C}_{\mathrm{triv}}.$$

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Proposition

One has
$$\mathcal{O}_q(G) \cong U_q(\mathfrak{g})^{\circ}_{\text{type I}}, \quad \mathcal{O}_q(G/L_S) \cong C_q(\mathfrak{g}/\mathfrak{l}_S)^{\circ}$$
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- Idea: Study codifferential calculi on $C_q(\mathfrak{g}/\mathfrak{l}_S)$ instead.

Theorem (Heckenberger–Kolb '07)

For an irreducible quantized flag manifold $\mathcal{O}_q(G/L_S)$, there exists an isomorphism

$$\Omega_q^{(0,\bullet)}(G/L_S) \cong \left(\underline{C_\bullet^S}\right)^\circ$$

where $C_\bullet^S$ denotes the parabolic BGG resolution of the trivial $U_q(\mathfrak{g})$-module (a similiar isomorphism exists for the holomorphic complex).

- The first term of $\underline{C_\bullet^S}$ is precisely $C_q(\mathfrak{g}/\mathfrak{l}_S)$.

First Order Codifferential Calculi

Definition

Let C be a coalgebra over $\mathbb{k}$, and let $\mathscr{W}_1$ be a C-bicomodule

- A $\mathbb{k}$-linear map $\delta\colon \mathscr{W}_1 \to C$ is called *coderivation* if

$$\Delta(\delta(w)) = \delta(w_{(0)}) \otimes w_{(1)} + w_{(-1)} \otimes \delta(w_{(0)})$$

  for all $w \in \mathscr{W}_1$.
- Let $\delta\colon \mathscr{W}_1 \to C$ be a coderivation. The pair $(\mathscr{W}_1, \delta)$ is called *first order codifferential calculus (focc)* if the map

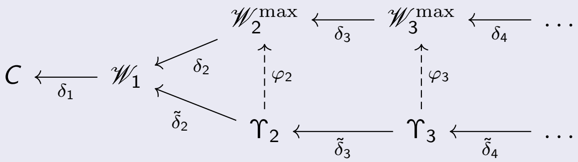
$$\mathscr{W}_1 \to C \otimes C, \quad w \mapsto w_{(-1)} \otimes \delta(w_{(0)})$$

  is injective.

Theorem (B '25)

Let $(\mathscr{W}_1, \delta)$ be a first order codifferential calculus over C. Then there exists a dg coalgebra $(\mathscr{W}_\bullet^{\max}, \delta_\bullet)$ with the following properties:

1. $\mathscr{W}_0 = C, \mathscr{W}_1^{\max} = \mathscr{W}_1$ and $\delta = \delta_1 \colon \mathscr{W}_1^{\max} \to C$.
2. For any other dg coalgebra $(\Upsilon_\bullet, \tilde{\delta}_\bullet)$ satisfying the first property, the identity $\mathrm{id}_{C \oplus \mathscr{W}_1}$ extends uniquely to a morphism of dg coalgebras $\varphi \colon \Upsilon_\bullet \to \mathscr{W}_\bullet^{\max}$.



Definition

The pair $(\mathscr{W}_\bullet, \delta_\bullet)$ is called maximal prolongation of $(\mathscr{W}_1, \delta)$.

- Note: $C_q(\mathfrak{g}/\mathfrak{l}_S)$ is a module coalgebra over $U_q(\mathfrak{g})$, and by a result of Masuoka, $U_q(\mathfrak{g})$ is faithfully flat over $U_q(\mathfrak{l}_S)$.

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- From now on: $H \subseteq U$ faithfully flat inclusion of Hopf algebras, and

$$C := U \otimes_H \mathbb{k}.$$

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- Denote by ${}^C_U\mathcal{M}^C$ respectively ${}_H\mathcal{M}^C$ the categories of C-bicovariant left U-modules, respectively right C-covariant left H-modules.

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A focc $(\mathscr{W}_1, \delta)$ is *U-equivariant* if $\mathscr{W}_1 \in {}^C_U\mathcal{M}^C$ and δ is U-linear.

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Proposition

Let $(\mathscr{W}_1, \delta)$ be a U-equivariant focc, then also $\mathscr{W}_\bullet^{\max} \in {}^C_U\mathcal{M}^C$ and $\delta_\bullet$ is U-linear.

Hermisson-Heckenberger–Kolb Classification for FOCC's

Definition

A *quantum tangent space* is a subspace $T \subseteq C^+ = \ker(\varepsilon)$ such that $HT \subseteq T$ and $\Delta(T) \subseteq (T \oplus \mathbb{k}[1]) \otimes C$.

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Remark

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Theorem (B '25)

There is a 1:1-correspondence

$$\{U\text{-equivariant focc's}\} \overset{1:1}{\leftrightarrow} \{\text{quantum tangent spaces } T \subseteq C^+\}$$

given by $(\mathscr{W}_1, \delta) \mapsto {}^{\mathrm{co}\, C}\mathscr{W}_1$ and $T \mapsto (U \otimes_H T, \delta)$ with $\delta(x \otimes_H t) = xt$. Moreover, the universal focc corresponds to $U \otimes_H C^+$.

Theorem (B '25)

Let $T \subseteq C^+$ be a quantum tangent space, such that the right C-coaction is given by $t \mapsto t \otimes [1]$ for all $t \in T$. Let $\mathscr{W}_1 := U \otimes_H T$ be the corresponding first order codifferential calculus. Consider the map

$$\hat{\delta}\colon T \otimes T \to U \otimes_H C^+, \quad [x] \otimes [y] \mapsto x_{(1)} \otimes_H [S(x_{(2)})y]$$

and let $R := \hat{\delta}^{-1}(U \otimes_H T) \subseteq T \otimes T$. Then the maximal prolongation can be written as

$$\mathscr{W}_\bullet^{\max} \cong U \otimes_H C(T, R)$$

where $C(T, R)$ is the quadratic coalgebra given by the quadratic datum $T, R \subseteq T \otimes T$:

$$C(T, R) := \Bbbk \oplus T \oplus R \oplus \bigoplus_{n \geq 3} \bigcap_{i+j+2=n} T^{\otimes i} \otimes R \otimes T^{\otimes j} \subseteq T^c(T).$$

The Podleś Cocalculus

- Let $U := U_q(\mathfrak{sl}_2)$ generated by E, F, K, K^{-1} and let $H = U_q(\mathfrak{l}_S)$ be the subalgebra generated by K and K^{-1}.

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- Then the maximal prolongation of the corresponding focc can be written as

$$
\begin{array}{ccccc}
 & & U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{l}_S)} \mathbb{C}_2 & & \\
 & \overset{F}{\nearrow} & & \overset{E}{\searrow} & \\
U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{l}_S)} \mathbb{C}_{\mathrm{triv}} & & \bigoplus & & U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{l}_S)} \mathbb{C}_{triv} \\
 & \underset{-E}{\searrow} & & \underset{F}{\nearrow} & \\
 & & U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{l}_S)} \mathbb{C}_{-2} & &
\end{array}
.
$$

The Antiholomorphic Cocalculus on $\mathcal{O}_q(\mathbb{C}\mathrm{P}^r)$

- Now let $U = U_q(\mathfrak{sl}_{r+1})$ and $H = U_q(\mathfrak{l}_S)$ with $S = \{1, \ldots, r\} \setminus \{1\}$.

The Antiholomorphic Cocalculus on $\mathcal{O}_q(\mathbb{CP}^r)$

- Now let $U = U_q(\mathfrak{sl}_{r+1})$ and $H = U_q(\mathfrak{l}_S)$ with $S = \{1, \dots, r\} \setminus \{1\}$.
- Consider the quantum tangent space $T^{(0,1)} \subseteq C_q(\mathfrak{g}/\mathfrak{l}_S)$ given by

$$\mathrm{span}_{\mathbb{C}}\{[E_1], [E_2 E_1], \dots, [E_r E_{r-1} \dots E_1]\}.$$

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- The maximal prolongation of $U_q(\mathfrak{sl}_{r+1}) \otimes_{U_q(\mathfrak{l}_S)} T^{(0,1)}$ can be written as

$$U_q(\mathfrak{sl}_{r+1}) \otimes_{U_q(\mathfrak{l}_S)} \Lambda_q^{\bullet,c}(T^{(0,1)})$$

where $\Lambda_q^{\bullet,c}(T^{(0,1)})$ is the dual coalgebra of

$$\Lambda_q^{\bullet}(T^{(0,1)}) = T(T^{(0,1)})/\langle e_i \otimes e_j + q^{-1} e_j \otimes e_i \mid 1 \leq i, j \leq r \rangle.$$

$(e_i := [E_i E_{i-1} \ldots E_1])$

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