Hyperbolic hypergeometric functions

Martin Hallnäs

Chalmers University of Technology
University of Gothenburg

Integrable systems: regularity, non-commutativity and random matrix theory
International Centre for Mathematical Sciences
15-19 June 2026

Overview

1. Univariate hypergeometric functions

2. Multivariate hypergeometric functions

Univariate hypergeometric functions

Classical hypergeometric theory

Gauss hypergeometric function/series:

$$ {}_2F_1\left(\begin{matrix} a, & b, \\ & c \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n, $$

where $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$.

Hypergeometric differential equation:

$$ z(1-z)_2F_1'' + (c - (a+b+a)z)_2F_1' - ab_2F_1 = 0. $$

Rmks:

- Assuming $c \notin \mathbb{Z}$, it is the unique solution analytic at $z = 0$ having the normalisation

$$ {}_2F_1\left(\begin{matrix} a, & b, \\ & c \end{matrix} \middle| 0\right) = 1. $$

- Any 2nd order ODE w/ three regular singular points can be transformed into the hypergeometric diff. eq. by a Möbius transformation $z = (\alpha x + \beta)/(\gamma x + \delta)$.

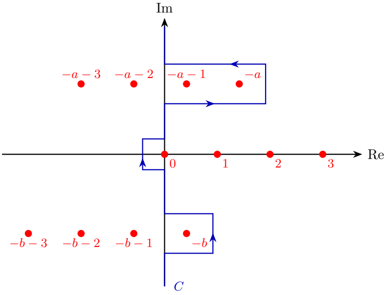
Classical hypergeometric theory

Barnes' integral representation:

$$ {}_2F_1\left(\begin{matrix} a,\ b, \\ c \end{matrix}\,\middle|\, z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_C \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \frac{ds}{2\pi i} $$

Series and diff. eq. can be obtained from the following properties of the Gamma function Γ:

1. $\Gamma(z)$ is meromorphic,
2. w/ simple poles at $z = 0, -1, -2, \ldots$,
3. $\Gamma(z+1) = z\Gamma(z)$.



Basic hypergeometric theory

Recall:

- q-Pochhammer symbol

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

requiring $|q| < 1$ when $n = \infty$.

- q-Gamma function

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}$$

Rmks:

- $\Gamma_q(z)$ only well-defined when $|q| < 1$,
- $\lim_{q\to 1^-} \Gamma_q(z) = \Gamma(z)$,
- $\Gamma_q(z)$ is meromorphic with simple poles at $z = -n \pm 2\pi i m/\log q$ ($n, m \in \mathbb{Z}_{\geq 0}$) and no zeros,
- $\Gamma_q(z)$ is uniquely characterised by

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z), \quad \Gamma_q(0) = 1$$

and logarithmic convexity (cf. the Bohr–Möllerup thm.).

Basic hypergeometric theory

Heine's basic hypergeometric function/series:

$$ {}_2\phi_1\left(\begin{matrix} a, & b, \\ & c \end{matrix}\,\middle|\, q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n. $$

Barnes type integral representation:

$$ {}_2\phi_1\left(\begin{matrix} q^a, & q^b, \\ & q^c \end{matrix}\,\middle|\, q, z\right) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)} \int_C \frac{\Gamma_q(a+s)\Gamma_q(b+s)}{\Gamma_q(c+s)\Gamma_q(s+1)} \frac{-\pi}{\sin(\pi s)} (-z)^s \frac{ds}{2\pi i} $$

q-difference equation:

$$ z(c - q^{1+a+b}z) D_q^2\, {}_2\phi_1 + \left(\frac{1-q^c}{1-q} + \frac{(1-q^a)(1-q^b) - (1-q^{1+a+b})}{1-q} z\right) D_q\, {}_2\phi_1 $$
$$ = \frac{(1-q^a)(1-q^b)}{(1-q)^2}\, {}_2\phi_1, \quad D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}. $$

The concept of doubling

From van de Bult's (2007) thesis "Hyperbolic hypergeometric functions".

Example: $f(x) = 1/(x;q)_\infty$ which satisfies

$$f(qx) = (1-x)f(x), \quad 0 < |q| < 1.$$

Additive reformulation:

- Substitute

$$x = e^{2\pi z/\omega_-}, \quad q = e^{2\pi i\omega_+/\omega_-}$$

  with

$$\operatorname{Re}\omega_\pm > 0, \quad \omega_+/\omega_- \in \mathbb{H}.$$

- The resulting meromorphic function

$$g(z) = 1\big/\big(e^{2\pi z/\omega_-}; e^{2\pi i\omega_+/\omega_-}\big)_\infty = 1\Big/\prod_{k=0}^{\infty}\big(1 - e^{2\pi(z+ik\omega_+)/\omega_-}\big)$$

  satisfies two difference eqs.:

$$g(z+i\omega_+) = \big(1 - e^{2\pi z/\omega_-}\big)g(z),$$
$$g(z+i\omega_-) = g(z).$$

The concept of doubling

- Consider the “doubled” or $\omega_+ \leftrightarrow \omega_-$-invariant system:

$$g(z+i\omega_+) = \left(1 - e^{2\pi z/\omega_-}\right) g(z),$$
$$g(z+i\omega_-) = \left(1 - e^{2\pi z/\omega_+}\right) g(z).$$

- An obvious solution:

$$g(z) = \frac{\left(e^{2\pi(z-i\omega_-)/\omega_+}; e^{-2\pi i\omega_-/\omega_+}\right)_\infty}{\left(e^{2\pi z/\omega_-}; e^{2\pi i\omega_+/\omega_-}\right)_\infty},$$

  where $-\omega_-/\omega_+ = -1/(\omega_+/\omega_-) \in \mathbb{H}$.

- $g(z) \equiv g(\omega_+, \omega_-; z)$ has a meromorphic extension to all $\mathrm{Re}\,\omega_\pm > 0$ and $z \in \mathbb{C}$, which includes

$$\omega_+/\omega_- \in \mathbb{R}_{>0} \Rightarrow |q| = \left|e^{2\pi i\omega_+/\omega_-}\right| = 1.$$

- This is essentially Ruijsenaars' (1997) hyperbolic Gamma function:

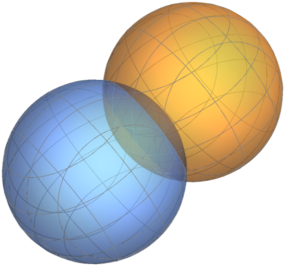
$$G(\omega_+, \omega_-; z) \equiv \exp\left(i\int_0^\infty \left(\frac{\sin 2yz}{2\sinh(\omega_+ y)\sinh(\omega_- y)} - \frac{z}{\omega_+\omega_- y}\right)\frac{dy}{y}\right)$$

  for $\mathrm{Im}\, z < (\omega_+ + \omega_-)/2$.

Hyperbolic hypergeometric functions

Example: Recall the conical function specialisation

$$F(g;r,k) \equiv {}_2F_1\left(\begin{matrix} g+ik,\ g-ik, \\ g+1/2 \end{matrix} \,\middle|\, -\sinh^2(r/2)\right)$$
$$= C(g;k)\int_{\mathbb{R}} \prod_{\delta_1,\delta_2=\pm} \Gamma((\delta_1 it+\delta_2 ik+g)/2)e^{itr}\,dt.$$



Introduced and studied by Mehler (1868) when solving the Dirichlet problem on a domain bounded by two intersecting spheres.

Ruijsenaars (2011) introduced a hyperbolic/relativistic generalisation:

$$J(b;x,y) \equiv \int_{\mathbb{R}} \prod_{\delta_1,\delta_2=\pm} G(\delta_1 z+\delta_2 x/2-ib/2)e^{\frac{2\pi i}{\omega_+\omega_-}zy}\,dz.$$

Rmks:

- Special case of Ruijsenaars' (1999) 8-variable R-function, given by a generalisation of Barnes' representation for ${}_2F_1$.
- As observed by van de Bult, Rains & Stokman (2007), the R-function is a degeneration of Spiridonov's (2005) hyperbolic hypergeometric equation.

Hyperbolic hypergeometric functions

A doubled system of difference eqs.:

$$\frac{\sinh\left(\frac{\pi}{\omega_+}(z-ib)\right)}{\sinh\left(\frac{\pi}{\omega_+}z\right)}J(z+i\omega_-,y)+\frac{\sinh\left(\frac{\pi}{\omega_+}(z+ib)\right)}{\sinh\left(\frac{\pi}{\omega_+}z\right)}J(z-i\omega_-,y)=2\cosh\left(\frac{\pi}{\omega_+}y\right)J(z,y)$$

$$\frac{\sinh\left(\frac{\pi}{\omega_-}(z-ib)\right)}{\sinh\left(\frac{\pi}{\omega_-}z\right)}J(z+i\omega_+,y)+\frac{\sinh\left(\frac{\pi}{\omega_-}(z+ib)\right)}{\sinh\left(\frac{\pi}{\omega_-}z\right)}J(z-i\omega_+,y)=2\cosh\left(\frac{\pi}{\omega_-}y\right)J(z,y)$$

Rmks:

- Doubled system of eigenvalue equations for A_1 Macdonald–Ruijsenaars operator.
- Setting $(\omega_+,\omega_-,b,y)=(\pi,\beta,\beta g,\beta k)$ and taking $\beta\to 0$, the first eq. becomes trivial and the second an eigenvalue equation for the A_1 hyperbolic Calogero–Moser operator

$$L\equiv-\frac{d^2}{dz^2}+2g\coth z\,\frac{\partial}{\partial z}.$$

Multivariate hypergeometric functions

Classical- and basic cases

- Heckman & Opdam (1987) introduced a hypergeometric function F_R associated w/ a root system R.
- They showed that F_R is the unique solution at $x = 0 \in \mathbb{C}^r$, $r = \operatorname{rank} R$, of a system of eigenvalue equations for an integrable system of Calogero–Moser operators.
- When $R = A_{N-1}$, the relevant operators are

$$L_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2g \sum_{1 \leq j < k \leq N} \coth \frac{x_j - x_k}{2} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right)$$

  and higher order operators commuting w/ L_N.
- In the basic case, there is Cherednik's (1997) global spherical function.
- By obtaining its c-function expansion, Stokman (2014) showed that it is a natural q-extension of the Heckman-Opdam function F_R.

Macdonald–Ruijsenaars operators

Integrable system of q-difference operators:

$$D_N^r(x) = t^{\binom{r}{2}} \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i} \quad (r = 1,\dots,N)$$

where

$$(T_{q,x_i} f)(x) = f(x_1 \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_N).$$

Rmks:

- From Ruijsenaars (1987) and Macdonald (1988), we know they commute and are simultaneously diagonalised by Macdonald polynomials.
- To recover operators in conical case, set

$$N = 2, \quad x_1/x_2 = e^{2\pi z/\omega_\mp}, \quad q = e^{2\pi i\omega_\pm/\omega_\mp}, \quad t = e^{-2\pi i b/\omega_\mp}.$$

Basic Harish–Chandra series

For each matrix $\theta \in M_N = \left\{\theta = (\theta_{ij})_{i,j=1}^N \in \mathrm{Mat}(N;\mathbb{N}) \mid \theta_{ij} = 0 \text{ if } 1 \le j \le i \le N\right\}$, introduce a rational function

$$c_N(\theta, s \mid q, t) = \prod_{k=2}^{N} \prod_{1\le i<j\le k} \frac{\left(q^{\sum_{a>k}(\theta_{ia}-\theta_{ja})} t s_j/s_i; q\right)_{\theta_{ik}}}{\left(q^{\sum_{a>k}(\theta_{ia}-\theta_{ja})} q s_j/s_i; q\right)_{\theta_{ik}}} \cdot \prod_{1\le i\le j<k} \frac{\left(q^{-\theta_{jk}+\sum_{a>k}(\theta_{ia}-\theta_{ja})} q s_j/t s_i; q\right)_{\theta_{ik}}}{\left(q^{-\theta_{jk}+\sum_{a>k}(\theta_{ia}-\theta_{ja})} s_j/s_i; q\right)_{\theta_{ik}}}.$$

Consider the “asymptotically free” power series

$$p_N(x, s \mid q, t) = \sum_{\theta\in M_n} c_N(\theta, s \mid q, t) \prod_{1\le i<j\le N} (x_j/x_i)^{\theta_{ij}}$$

with

$$p_2(x, s \mid q, t) = {}_2\phi_1\left(\begin{matrix} t,\ ts_2/s_1, \\ qs_2/s_1 \end{matrix} \,\middle|\, q, qx_2/tx_1\right).$$

Basic Harish–Chandra series

Noumi & Shiraishi (2012):

Writing $s_i = t^{n-i} q^{\lambda_i}$, we have

$$D_N^r(x) x^\lambda p_N(x, s \mid q, t) = x^\lambda p_N(x, s \mid q, t) \sum_{1 \leq i_1 < \cdots i_r \leq N} s_{i_1} \cdots s_{i_r} \quad (r = 1, \ldots, N).$$

Rmks:

- $x^\lambda p_N(x, s \mid q, t) \sim x^\lambda$ deep in the Weyl chamber $x_N < \cdots < x_1$.
- Such basic Harish–Chandra series have been studied from various perspectives by Cherednik (1992), Etingof & Kirillov (1994), Stokman & collaborators (Letzter, van Meer),...
- When $t = q^{-m}$, $m \in \mathbb{Z}_{>0}$, we essentially get Chalykh's (2002) Baker–Akhiezer function for the Macdonald–Ruijsenaars operators D_N^r.

Doubling Macdonald–Ruijsenaars operators

Taking

$$x_i = e^{\frac{2\pi}{\omega_\mp} z_i}, \quad q = q_\pm := e^{-2\pi i \omega_\pm / \omega_\mp}, \quad t = e^{-\frac{2\pi i}{\omega_\mp} b}$$

we get

$$D_N^r = \text{const.} \underbrace{\sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{\sinh\left(\frac{\pi}{\omega_\mp}(z_i - z_j - ib)\right)}{\sinh\left(\frac{\pi}{\omega_\mp}(z_i - z_j)\right)} \prod_{i \in I} e^{-i\omega_\pm \partial_{z_i}}}_{=: A_{N,\mp}^r}.$$

Since the coefficients in the right-hand side are $i\omega_\mp$-periodic, the $2N$ difference operators $A_{N,\mp}^r$ commute.

Doubled system of Macdonald–Ruijsenaars eigenvalue equations:

$$A_{N,\pm}^r(x) J(x,y) = J(x,y) \sum_{1 \leq i_1 < \cdots i_r \leq N} e^{\frac{\pi}{\omega_\mp}(y_{i_1} + \cdots + y_{i_r})} \quad (r = 1, \ldots, N)$$

The hyperbolic case

Consider the generalized Harish–Chandra c-function

$$c(b;z)=\frac{G(z+i(\omega_++\omega_-)/2-ib)}{G(z+i(\omega_++\omega_-)/2)},\quad C_N(b;z)=\prod_{1\leq j<k\leq N}c(b;z_j-z_k).$$

Conjecture:

Let

$$q_\delta=e^{-2\pi i\omega_\delta/\omega_{-\delta}},\quad t_\delta(b)=q_\delta^{b/a_\delta},\quad \delta=+,-$$

and $\rho_N=(1/2)(N-1,N-3,\dots-N+3,-N+1)$. Then

$$J_N(b;x,y)\equiv\sum_{\sigma\in S_N}C_N(b;\sigma(y))e^{2\pi i(x,\sigma(y)+ib\rho_N)}$$

$$\cdot\prod_{\delta=+,-}p_N(e_{-\delta}(2x),e_{-\delta}(2\sigma(y))\mid q_\delta,t_\delta(b))$$

is a solution to the doubled system of Macdonald–Ruijsenaars eigenvalue equations that is analytic for $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$. Moreover, it extends to a meromorphic function on $\mathbb{C}^N\times\mathbb{C}^N$.

The hyperbolic case

Known cases:

- $N = 2$: van de Bult, Rains & Stokman (2007).
- $b = n\omega_\pm$: Basic Harish–Chandra series p_N equals Chalykh's (2002) Baker–Akhiezer function for Macdonald–Ruijsenaars operators.

Proof idea:

1. From H. & Ruijsenaars (2014), we know an analytic solution around $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$:

$$J_N(x,y) = e^{\frac{2\pi i}{\omega_+\omega_-} y_N(x_1+\cdots+x_N)} \cdot \int_{\mathbb{R}^{N-1}} \frac{\prod_{j=1}^N \prod_{k=1}^{N-1} \prod_{\delta=\pm} G(\delta(x_j - z_k) - ib/2)}{C_{N-1}(z)C_{N-1}(-z)} \cdot J_{N-1}(z, (y_1 - y_N, \ldots, y_{N-1} - y_N))dz,$$

   with $J_1(x, y) = \exp(\frac{2\pi i}{\omega_+\omega_-} xy)$.

2. Shift contours $\mathbb{R} \to \mathbb{R} + i\eta$, taking $\eta \to \infty$, picking up residues in the process.
3. Use recursive structure of p_N, identified by Noumi & Shiraishi (2012) to implement a proof by induction.

The hyperbolic case

Partial results:

- Leading term: H. & Ruijsenaars (2019).
- $b = n\omega_{\pm}$: in this case, I recently used the above strategy to deduce the expansion of J_N in terms of Chalykh's (2002) Baker–Akhiezer function for the Macdonald–Ruijsenaars operators.

Reference:
H.: *Baker–Akhiezer specialisation of joint eigenfunctions for hyperbolic relativistic Calogero–Moser Hamiltonians.*
Math. Phys. Anal. Geom. 29 (2026).

UNIVERSITY OF
GOTHENBURG

CHALMERS
UNIVERSITY OF TECHNOLOGY