Lusztig differential calculi on the non-irreducible quantum flag manifolds

New perspectives in quantum representation theory.

ICMS, Edinburgh.

Antonio Del Donno

November 19th 2025

Differential structures over algebras

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Definition

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1. an A-bimodule Γ;
2. a linear map $\mathrm{d} : A \to \Gamma$ satisfying the Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ for every $a, b \in A$;
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A differential calculus on a $\mathbb{K}$-algebra A is a differential graded algebra $(\Omega^\bullet, \wedge, \mathrm{d})$ which is generated in degree zero and such that $\Omega^0 = A$. The former means that

$$\Omega^k = \mathrm{span}_{\mathbb{K}}\{a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k : a^0, \ldots, a^k \in A\}.$$

We call elements of Ω^k differential k-forms.

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Theorem

Let $(\Omega^\bullet, \widetilde{\wedge}, \widetilde{\mathrm{d}})$ be any differential calculus on A such that $\Omega^1 = \Gamma$ and $\widetilde{\mathrm{d}}|_A = \mathrm{d}$. There exists a surjective morphism $\Gamma^\bullet \to \Omega^\bullet$ of differential graded algebras. In particular, $(\Omega^\bullet, \widetilde{\wedge}, \widetilde{\mathrm{d}})$ is a quotient of $(\Gamma^\bullet, \wedge, \mathrm{d})$.

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Theorem (Woronowicz '89)

There is a bijective correspondence

$$\{\textit{left-covariant FODCi on } H\} \iff \{\textit{right ideals} \subseteq \ker \varepsilon\}.$$

The calculus is bicovariant if and only if the corresponding ideal $I \subseteq \ker \varepsilon$ is $\mathrm{Ad}-$*invariant, where* $\mathrm{Ad}(h) = h_2 \otimes S(h_1)h_3$.

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Let B as above. We say that B is a quantum homogeneous space if A is faithfully flat as a right B-module.

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We define the quantum flag manifold $\mathcal{O}_q(G/L_S)$ as the subalgebra

$$\mathcal{O}_q(G/L_s) := \{ b \in \mathcal{O}_q(G) \mid X \triangleright b = \varepsilon(X) b, \ \forall X \in U_q(\ell_S) \}.$$

of $U_q(\ell_S)$-invariant elements of $\mathcal{O}_q(G)$.

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- Quantum flag manifolds are examples of quantum homogeneous spaces.

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- $\Omega^1(B)$ is a left A-covariant B-bimodule, with bimodule structure

$$b(a \otimes [c])b' := b\,a\,b'_1 \otimes [c\,b'_2].$$

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Theorem (Hermisson, '02)

The pair $(\Omega^1(B), \mathrm{d})$ is a left A-covariant FODC on B. Moreover, every left A-covariant FODC on B is of this form. We call $V^1 := B^+/I$ the quantum cotangent space associated to $\Omega^1(B)$.

Quantum tangent spaces

- Let A be a Hopf algebra, W a Hopf subalgebra of A° such that

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Theorem (Heckenberger–Kolb, '03)

There is a bijective correspondence between isomorphism classes of finite-dimensional tangent spaces and finitely generated left A-covariant FODCi on B.

Irreducible flag manifolds

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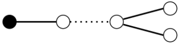
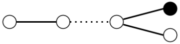
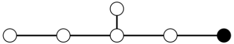
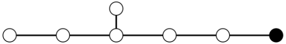
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{Q}_{2n})$
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{S}_n)$
E_6	$\mathcal{O}_q(E_6)$		$\mathcal{O}_q(\mathbb{OP}^2)$
E_7	$\mathcal{O}_q(E_7)$		$\mathcal{O}_q(\mathbf{F})$

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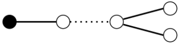
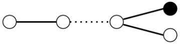
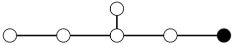
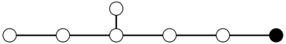
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{Q}_{2n})$
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{S}_n)$
E_6	$\mathcal{O}_q(E_6)$		$\mathcal{O}_q(\mathbb{OP}^2)$
E_7	$\mathcal{O}_q(E_7)$		$\mathcal{O}_q(\mathbf{F})$

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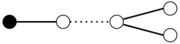
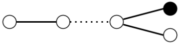
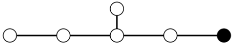
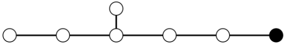
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{Q}_{2n})$
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{S}_n)$
E_6	$\mathcal{O}_q(E_6)$		$\mathcal{O}_q(\mathbb{OP}^2)$
E_7	$\mathcal{O}_q(E_7)$		$\mathcal{O}_q(\mathbf{F})$

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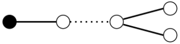
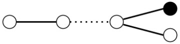
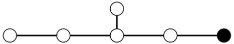
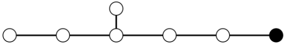
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{Q}_{2n})$
D_n	$\mathcal{O}_q(\mathrm{Spin}_{2n})$		$\mathcal{O}_q(\mathbf{S}_n)$
E_6	$\mathcal{O}_q(E_6)$		$\mathcal{O}_q(\mathbb{OP}^2)$
E_7	$\mathcal{O}_q(E_7)$		$\mathcal{O}_q(\mathbf{F})$

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- Question: do we have the same behaviour for the non-irreducible setting?

The action of the braid group on $U_q(\mathfrak{g})$

- A breakthrough result of Lusztig was to show that there is an action of the braid group on $U_q(\mathfrak{g})$:

$$B_{\mathfrak{g}} \to \mathrm{End}_{\mathrm{alg}}(U_q(\mathfrak{g})),$$
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Definition (Lusztig)

Let W be the Weyl group of $\mathfrak{g}$, w_0 the longest element of W, and $w = w_{i_1} \cdots w_{i_n}$ a choice of reduced decomposition of w_0. We consider

$$E_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(E_{i_r}), \qquad F_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(F_{i_r})$$

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- We can use the result of Lusztig to build tangent spaces and thus differential calculi.

The non-irreducible setting: A-series full flags

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- *The restriction of the maximal prolongation of the corresponding differential calculus to the full flag manifold $O_q(F_n)$ has classical dimension.*

The non-irreducible setting: a conjecture for the one crossed partial flags

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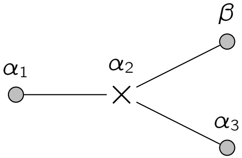

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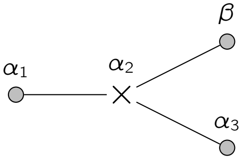

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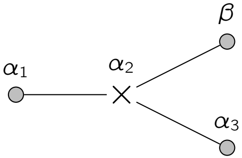

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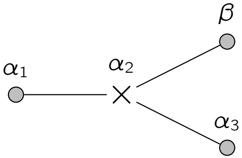

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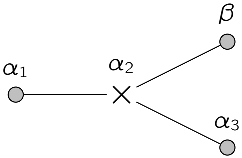

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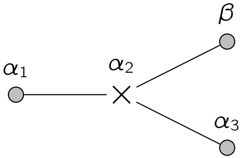

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- A parallel analysis is currently underway for $\mathfrak{g} = \mathfrak{sp}(4)$, with analogous partial results and expectations.