Noncommutative martingale inequalities: some recent progress I

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Connections between commutative
and non-commutative harmonic analysis

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Joint works with T. Gałązka, Y. Jiao, L. Wu and Y. Zuo

Plan

I. Inequalities for martingale transforms

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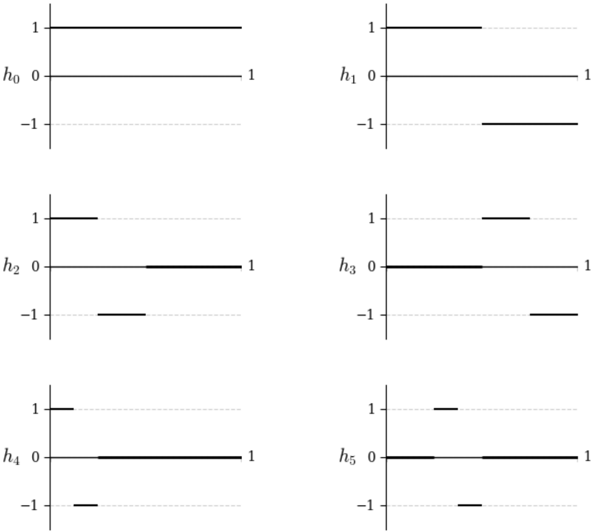
II. Weighted maximal inequalities

I. Inequalities for martingale transforms

Motivation: unconditionality of the Haar system $(h_n)_{n\geq 0}$



Haar system is an unconditional basis

Theorem (Schauder 1927)

The sequence $(h_n)_{n\geq 0}$ *is a basis of* L^p, $1 \leq p < \infty$: *for any* $f \in L^p$,

$$f = \sum_{n=0}^{\infty} a_n h_n \qquad (\textit{convergence in } L^p)$$

for some unique coefficients $a_0, a_1, a_2, \ldots$.

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Theorem (Marcinkiewicz-Paley 1932)

For any $1 < p < \infty$ there is a finite constant c_p such that

$$\left\| \sum_{n=0}^{N} \varepsilon_n a_n h_n \right\|_{L^p} \leq c_p \left\| \sum_{n=0}^{N} a_n h_n \right\|_{L^p}$$

for any N, $a_0, a_1, a_2, \ldots a_N \in \mathbb{R}$ and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \in \{-1, 1\}$.

A related classical result

For an integrable $f : \mathbb{T} \to \mathbb{C}$, consider the Fourier expansion

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Theorem (Riesz, 1926)

For any $1 < p < \infty$, *we have* $\|\mathcal{H}\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} < \infty$.

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- Define the associated differences

$$dx_0 = x_0, \quad dx_n = x_n - x_{n-1}, \quad n \geq 1 \quad \leftrightarrow \quad x_N = \sum_{n=0}^{N} dx_n.$$

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To guarantee the integrability of y one typically assumes $v \subset L^\infty(\mathcal{M})$.

L^p bounds for martingale transforms

Theorem (Burkholder 1966, classical case)

For any $1 < p < \infty$ *there is a finite constant* c_p *such that*

$$\left\| \sum_{n=0}^{N} v_n dx_n \right\|_p \leq c_p \left\| \sum_{n=0}^{N} dx_n \right\|_p, \qquad N = 0, 1, 2, \ldots,$$

if $x \in L^p$ *and the transforming sequence* v *is bounded by* 1.

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Motivates the introduction of UMD spaces.

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Burkholder, 1984: optimal choice for c_p is $\max\{p-1, (p-1)^{-1}\}$.

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Suppose that for each n, v_n belongs to the commutant $\mathcal{M}_n'$.

Theorem (Pisier-Xu 1997, noncommutative case)

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L^p bounds for martingale transforms

Suppose that for each n, v_n belongs to the commutant $\mathcal{M}'_n$.

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- Applications: Fourier multipliers, Stein inequalities, etc.

The off-diagonal, commutative case

Recall
$$\left\| \sum_{n=0}^{N} v_n dx_n \right\|_p \leq c_p \left\| \sum_{n=0}^{N} dx_n \right\|_p \cdot \sup_n \|v_n\|_\infty .$$

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$$\left\| \sum_{n=0}^{N} v_n dx_n \right\|_p \leq c_p \left\| \sum_{n=0}^{N} dx_n \right\|_p \cdot \left\| \sup_n |v_n| \right\|_\infty .$$

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Let q, $r > 1$ and set $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Theorem (Gałązka, O. 2023)

There is $c_{p,q,r} < \infty$ such that if $x \in L^q$ and $\sup_n |v_n| \in L^r$,

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We identified optimal constants $c_{p,q,r}$, studied weak-type bounds.

Noncommutative maximal function(s)

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Let $V = (V_n)_{n\geq 0}$ be defined by $V_0 = |v_0|$ and for $n \geq 1$,

$$V_n = \left(V_{n-1}^2 + \max\{|v_n|^2 - V_{n-1}^2, 0\}\right)^{1/2}.$$

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Lemma

- *We have* $|v_n|^2 \leq V_n^2 \leq V_{n+1}^2$ *for each* n.
- *If* v *is predictable, so is* V.
- V^2 *majorizes* $|v^*|^2$, $|\Re v|^2$ *and* $|\Im v|^2$.

The off-diagonal, noncommutative case

Theorem (O. 2025)

Suppose that x is a martingale bounded in L^q, v is a predictable sequence bounded in L^r such that $v_n \in \mathcal{M}_n'$ for each n. Then

$$\left\| \sum_{n=0}^{N} v_n dx_n \right\|_{L^p(\mathcal{M})} \leq c_{p,q,r} \left\| \sum_{n=0}^{N} dx_n \right\|_{L^q(\mathcal{M})} \cdot \|V_N\|_{L^r(\mathcal{M})}, \qquad N \geq 0.$$

· *If $p > 2$, then $c_{p,q,r}$ can be taken dependent on p only, and $c_{p,q,r} = O(p)$ as $p \to \infty$ (optimal).*

· *If $q < 2$, it can be taken dependent on q only and $c_{p,q,r} = O((q-1)^{-1})$ as $q \to 1^+$ (optimal).*

· *If $p \leq 2 \leq q$, then one can take $c_{p,q,r} = 1$, which is sharp.*

A comment

Is the result related to fractional estimates? For the Haar system

$$\left\| \sum_{n=0}^{N} 2^{-\tilde{n}\alpha} a_n h_n \right\|_p \leq c_{p,q} \left\| \sum_{n=0}^{N} a_n h_n \right\|_q, \qquad \frac{1}{q} = \frac{1}{p} + \alpha.$$

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In a sense, our result goes in the other direction.

Let S_N^p stand for the Schatten class equipped with the trace Tr.

Theorem

Let $a_j \in \mathbb{C}$ with $|a_1| \geq |a_2| \geq \ldots \geq |a_N|$. Then

$$m = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_N \\ a_2 & a_2 & a_3 & \ldots & a_N \\ a_3 & a_3 & a_3 & \ldots & a_N \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_N & a_N & a_N & \ldots & a_N \end{pmatrix}$$

is a Schur multiplier between S_N^q and S_N^p:

$$\|m * x\|_{S_N^p} \leq c_{p,q,r} \|x\|_{S_N^q} \left(\sum_{n=1}^{N} |a_n|^r \right)^{1/r}.$$

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- We have $\|T_m\|_{L^2(G)\to L^2(G)} = \|m\|_{L^\infty}$ (trivial); other p?

Assume that G is equipped with a conditionally negative length function $\psi : G \to [0, \infty)$:

$$\psi(e) = 0, \quad \psi(g) = \psi(g^{-1}) \quad \text{for all } g \in G,$$

and we have

$$\sum_{g,h \in G} \overline{a_g} a_h \, \psi(g^{-1}h) \leq 0$$

holds for any sequence $(a_g)_{g \in G} \subset \mathbb{C}$ such that $\sum_{g \in G} a_g = 0$.

Let $(a_u)_{u\geq 0}$ be a commutative process and consider the function

$$A(s) = s\int_0^\infty e^{-2us}\mathbb{E}a_u du, \qquad s \geq 0.$$

Theorem

Consider the symbol $m(g) = A(\psi(g))$. Then we have

$$\|T_m\|_{L^q(G)\to L^p(G)} \leq c_{p,q,r}\left\|\sup_{u\geq 0}|a_u|\right\|_r.$$

II. Weighted maximal inequalities

Let $\mathcal{D}$ be the collection of dyadic subcubes of $[0,1)^d$.

Maximal operators

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For an integrable $f : [0,1)^d \to \mathbb{R}$, define its maximal function

$$Mf(x) = \sup\left\{ \left| \fint_Q f \right| \,:\, x \in Q,\ Q \in \mathcal{D} \right\}.$$

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We have the sharp bounds

$$\begin{aligned} \|Mf\|_{L^{p,\infty}} &\leq \|f\|_{L^p}, && 1 \leq p < \infty, \\ \|Mf\|_{L^p} &\leq \frac{p}{p-1}\|f\|_{L^p}, && 1 < p < \infty, \\ \|Mf\|_{L^{p,\infty}} &\leq \frac{p}{p-1}\|f\|_{L^{p,\infty}}, && 1 < p < \infty. \end{aligned}$$

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Huge literature on the subject, numerous methods (interpolation, covering lemmas, Bellman function approach, linearization, combinatorial arguments, etc.)

Weighted spaces

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For $1 < p < \infty$, w satisfies Muckenhoupt's condition A_p if

$$[w]_{A_p} := \sup_{Q \in \mathcal{D}} \left(\fint_Q w \right) \left(\fint_Q w^{1/(1-p)} \right)^{p-1} < \infty.$$

Two results

Theorem (Muckenhoupt '72)

Fix $1 < p < \infty$ *and a weight* w*. Then* $\|M\|_{L^p(w)\to L^p(w)} < \infty$ *if and only if* $w \in A_p$.

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For $1 < p < \infty$ and $w \in A_p$, we have the estimate

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Motivation: regularity results for solutions of certain PDEs.

Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra.

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Maximal inequalities for operator-valued functions

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However, the martingale maximal operator does not make sense:

$$\sup \left| \fint_Q f \right| = ???.$$

Theorem (Junge '01)

Fix $1 < p < \infty$. For any $f \in L^p_+$, there is a function $a \in L^p_+$ with

$$\int_Q f \leq a(x) \qquad \text{for all } Q \in \mathcal{D} \text{ and all } x \in Q,$$

and

$$\|a\|_{L^p} \leq C \left(\frac{p}{p-1}\right)^2 \|f\|_{L^p}$$

for some universal C.

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Theorem (Gałązka, Jiao, O., Wu '24)

Fix $1 < p < \infty$. For any weight $w \in A_p$ and any $f \in L^p_{w+}$, there is a majorant $a \in L^p_{w+}$ such that

$$\|a\|_{L^p_w} \leq C \left(\frac{p}{p-1} \right)^2 [w]_{A_p}^{1/(p-1)} \|f\|_{L^p_w}.$$

The exponent $1/(p-1)$ is the best possible.

A dual version

Theorem

Let $1 < p < \infty$ and $w \in A_p$. For any sequence $(a_n)_{n \geq 0}$ of functions from $[0,1)^d$ to $\mathcal{M}_+$ we have

$$\left\| \sum_{n \geq 0} \mathcal{E}_n(a_n) \right\|_{L^p_w} \lesssim p^2 [w]_{A_p} \left\| \sum_{n \geq 0} a_n \right\|_{L^p_w}$$

and the linear dependence on $[w]_{A_p}$ is optimal.

Applications

· Maximal inequalities on metric spaces;

· Ergodic theorems, concerning the action of amenable groups on noncommutative L^p spaces (in the spirit of Hong, Liao and Wang 2021);

· Weighted inequalities for some classes of maximal singular integrals.

The last slide...

Thank you for your attention!