Discrete & Quantum Dubrovin Equations

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Classical continuous Dubrovin Equations:

The *Dubrovin equations*[1] are the equations of motion in terms of the *auxiliary spectrum* in finite-gap integration theory.
In fact, they can be traced back to the seminal work by S. Kowalevski in her study of tops integrable through $g = 2$ Abelian functions.
They are almost *universal* (though mildly dependent on the model) they typically read:

$$\dot{\mu}_i = \frac{P(\mu_i)\sqrt{R(\mu_i)}}{\prod_{j\neq i}(\mu_i - \mu_j)} \qquad , \qquad (i = 1, \ldots, g),$$

(where $P(\mu)$ is some factor depending on the model).

The variables μ_i (auxiliary spectrum) are obtained as dynamical poles of the Baker-Akhiezer function (following recipe designed by P van Moerbeke (1976) and H Flaschka & McLaughlin (1976)) and led to effectively a separation of variables mechanism of the corresponding integrable system (e.g. KdV).
They can be linearised on the Jacobian of the hyperelliptic curve (of genus g)

$$y^2 = R(x) \quad , \quad (R \text{ polynomial of order } 2g+1 \text{ or } 2g+2),$$

through the *Abel map*.

[1] B.A. Dubrovin, *The periodic problem for the Korteweg-de Vries equation in the class of finite-gap potentials*, Funct. Anal. Appl. **9** (1975) 51;
B.A. Dubrovin, V.B. Matveev and S.P. Novikov, *Non-linear Equations of Korteweg-de Vries Type, Finite-zone Linear Operators, and Abelian Varieties*, Russ. Math. Surv. **31** (1976) 59–146.

Abstract

1. Classical KdV mappings and its (non-ultralocal) r-matrix structure
2. Classical discrete & continuous Dubrovin equations[2]
3. Quantization of integrable mappings of KdV type & quantum invariants (quantum curves)[3]
3. Quantum Dubrovin equations and reconstruction formulae[4] variables

[2] FWN, *Discrete Dubrovin Equations and Separation of Variables for Discrete Systems*, Chaos, Solitons and Fractals vol **11** (2000) 19–28.

[3] FWN & H.W. Capel, *Quantization of Integrable Mappings*, Springer Lecture Notes in Physics vol. **424** (1993) pp. 187–211.

[4] C M Field & F W Nijhoff, *Quantum discrete Dubrovin equations*, J. Phys. A: Math. Gen. **37** (2004) 8065.

Some ancient and more recent history:

Quantum discrete-time systems & quantum mappings in the non-integrable case have been considered by several people: M. Berry (1979), M. Tabor (1983), C.M. Bender et al. (1986).

The *integrable* discrete-time quantum case was developed in the early 1990s:

- L Faddeev & A Yu Volkov (1992)
  – QISM on space-time lattice;
- F W Nijhoff, HW Capel & V Papageorgiou (1992)
  – integrable quantum maps & extended YB structure for non-ultralocal discrete models;
- N. Reshetikhin, V. Bazhanov, A. Bobenko & N. Kutz (1994)
  – (quantum) group-theoretical description.
- V Pasquier & M Gaudin (1992), V Kuznetsov, E Sklyanin & collabs (1998), Derkachov (1999)
  – Quantum BTs and Q-Baxter operator.

Most early treatments dealt with the example of the *quantum Hirota map* (quantum lattice MKdV).

Discrete-time systems have appeared naturally (but only implicitely) as exactly solvable spin systems.

In later years the theory was developed further (Bazhanov et al., Korff, etc.) but quantum discrete Dubrovin eqs. have only appeared (to my knowledge) in our 2005 paper.

$*$ The main driver is to attempt to regularise the quantum systems which classically possess <u>singularities</u>.

A motivating example

Integrable quantum maps were first introduced in[5] The most general form of a quantum map 'of the plane' takes the form[6] given by the unitary operator:

$$U = e^{-\frac{i}{\hbar}T(p)}e^{-\frac{i}{\hbar}V(q)}$$

in terms of standard canonical operators p,q, obeying Heisenberg commutation relations: $[\, q\, ,\, p\,] = i\hbar$, and where

$$T(p) = \frac{1}{2}p^2 \quad , \qquad V(q) = q^2 + \frac{\beta q}{\alpha} + \frac{\alpha\epsilon - \beta^2}{2\alpha^2}\ln(\alpha q^2 + \beta q + \gamma)$$
$$+ \frac{2\alpha^2\zeta - 2\alpha\beta\gamma - \alpha\beta\epsilon + \beta^3}{\alpha^2\sqrt{4\alpha\gamma - \beta^2}}\tan^{-1}\left(\frac{2\alpha q + \beta}{\sqrt{4\alpha\gamma - \beta^2}}\right) .$$

The discrete operator equations of the motion

$$\widetilde{q} = U^\dagger q U = p - q - \frac{\beta q^2 + \epsilon q + \zeta}{\alpha q^2 + \beta q + \gamma}$$
$$\widetilde{p} = U^\dagger p U = \widetilde{q} - q \ ,$$

allow an exact quantum invariant (defining a quantum biquadratic curve):

$$\begin{aligned} I(\widetilde{q}, q) \quad &= \quad \frac{\alpha}{2}\left(q^2\widetilde{q}^2 + \widetilde{q}^2 q^2\right) + \frac{\beta}{2}\left(\widetilde{q}^2 q + q\widetilde{q}^2 + q^2\widetilde{q} + \widetilde{q}q^2\right) \\ &\quad + \gamma(q^2 + \widetilde{q}^2) + \frac{\epsilon}{2}(q\widetilde{q} + \widetilde{q}q) + \zeta(q + \widetilde{q}) \ . \end{aligned}$$

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[5] FWN, H.W. Capel & V.G. Papageorgiou, *Integrable Quantum Mappings*, Physical Review **A46** (1992)2155–2158.
G.R.W. Quispel & FWN, *Integrable 2-Dimensional Quantum Mappings*, Phys. Lett. **161A** (1992) 419–422.

[6] C. Field, *On the Quantization of Integrable Discrete-Time Systems*, PhD Thesis, University of Leeds (2005).

Remarks:

1.In contrast to continuous-time quantum systems, in discrete time the model is not given by its Hamiltonian, but rather through the operator equations of the motion (or through the unitary operator of the discrete time-evolution). In the integrable case these equations are rational expressions (classically allowing for the occurrence of singularities), and thus led to the question of their global well-posedness.

2.Even in the one-degree of freedom case, the existence of an exact quantum invariant is highly nontrivial. So far, quantum integrability criteria were focused on the existence of quantum separation of variables (cf. Sklyanin, 1991). Once separated, the role of integrable structures is no longer clear. However, in the discrete case there remains the issue of integrability *beyond separation*, namely already at the one-degree of freedom level.

3.The aim of this talk is to propose a possible route for making sense of the quantum discrete models, through an alternative representation of the operator equations of the motion, namely *in terms of the separated variables directly*. This is done through a quantum analogue of the discrete Dubrovin equations arising in the classical finite-gap theory.

Classical KdV mappings

The KdV mappings $\mathbf{R}^{2P} \to \mathbf{R}^{2P} : (\{v_j\}) \mapsto (\{\widetilde{v}_j\})$ are defined by

$$\widetilde{v}_{2j-1} = v_{2j} \ , \ \ \widetilde{v}_{2j} = v_{2j+1} + \frac{a}{v_{2j}} - \frac{a}{v_{2j+2}} \qquad (j = 1, \cdots, P),$$

where $a = p^2 - q^2$. Imposing the periodicity condition $v_{i+2P} = v_i$, it has Casimirs $\sum_{j=1}^{P} v_{2j} = \sum_{j=1}^{P} v_{2j-1} = C$. The local Lax representation is given by

$$L_j = V_{2j} \cdot V_{2j-1} \ , \ \ M_j = \begin{pmatrix} w_j & 1 \\ \lambda_{2j} & 0 \end{pmatrix} , \quad V_i = \begin{pmatrix} v_i & 1 \\ \lambda_i & 0 \end{pmatrix} , \text{s.t.} \boxed{\widetilde{L}_j M_j = M_{j+1} L_j}$$

in which $\lambda_{2j} = k^2 - q^2$, $\lambda_{2j+1} = k^2 - p^2$, $a = p^2 - q^2$, $w_j = v_{2j-1} - a/v_{2j}$.
Imposing Poisson relations $\{v_j \, , \, v_{j'}\} = \delta_{j+1,j'} - \delta_{j,j'+1}$ (with $\delta_{i,j}$ the Kronecker δ-symbol mod(2P)) we have the *non-ultralocal* relations

$$\{V_{n+1,1} \, , \, V_{n,2}\} = V_{n+1,1} \, s^{+}_{12}(n) \, V_{n,2} \ , \quad \{V_{n,1} \, , \, V_{n,2}\} = r^{+}_{12}(n) \, V_{n,1} V_{n,2} - V_{n,2} \cdot V_{n,1} \, r^{-}_{12} \ ,$$

and $\{V_{n,1} \, , \, V_{m,2}\} = 0$ for $|\, n - m| \geq 2$,
with r, s-matrices given by

$$r^{+}_{12} = r^{-}_{12} - s^{+}_{12} + s^{-}_{12} \ , \quad r^{-}_{12} = \frac{P_{12}}{\mu_1 - \mu_2} \ , \quad s^{+}_{12} = \frac{h}{\mu_2} F \otimes E \ , \quad s^{-}_{12} = s^{+}_{21} \ ,$$

in which $\mu_\alpha = k_\alpha^2 - q^2, \alpha = 1, 2$, P_{12} is the standard permutation operator and the matrices E and F are given by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Classical (extended) Yang-Baxter structure

The Lax matrices obey the following non-ultralocal Poisson relations

$$\{L_{n,1}, L_{m,2}\} = -\delta_{n,m+1}\, L_{n,1}\, s_{12}^{+}\, L_{m,2} \;+\; \delta_{n+1,m}\, L_{m,2}\, s_{12}^{-}\, L_{n,1}$$
$$+\,\delta_{n,m}\left[r_{12}^{+}\, L_{n,1} L_{m,2} \;-\; L_{n,1} L_{m,2}\, r_{12}^{-}\right] \;,$$

with $s^{\pm} = s^{\pm}(\lambda_1, \lambda_2)$ and $r^{\pm} = r^{\pm}(\lambda_1, \lambda_2)$subject to

$$s_{12}^{-}(\lambda_1, \lambda_2) \;=\; s_{21}^{+}(\lambda_2, \lambda_1) \;\;,\;\; r_{12}^{\pm}(\lambda_1, \lambda_2) \;=\; -r_{21}^{\pm}(\lambda_2, \lambda_1)\;.$$

The Jacobi identity of the Poisson brackets are satisfied if these obey the *classical Yang-Baxter relations*:

$$\begin{aligned}
\left[r_{12}^{\pm}\,,\, r_{13}^{\pm}\right] + \left[r_{12}^{\pm}\,,\, r_{23}^{\pm}\right] + \left[r_{13}^{\pm}\,,\, r_{23}^{\pm}\right] &= 0\,,\\
\left[s_{12}^{\pm}\,,\, s_{13}^{\pm}\right] + \left[s_{12}^{\pm}\,,\, r_{23}^{\pm}\right] + \left[s_{13}^{\pm}\,,\, r_{23}^{\pm}\right] &= 0.
\end{aligned}$$

Introducing the *monodromy matrix* $T(\lambda) \;\equiv \overleftarrow{\prod_{n=1}^{P}}\; L_n(\lambda)$, provided we have the relations $r_{12}^{+} - s_{12}^{+} \;=\; r_{12}^{-} - s_{12}^{-}$ we get the 'integrated' Poisson relations

$$\{T_1\,,\, T_2\} \;=\; r_{12}^{+} T_1 T_2 \;-\; T_2 T_1 r_{12}^{-} \;-\; T_1 s_{12}^{+} T_2 \;+\; T_2 s_{12}^{-} T_1\;.$$

Furthermore, under the mapping generated by the similarity transform $\widetilde{T}(\lambda) = M(\lambda) T(\lambda) M(\lambda)^{-1}$ (where $M := M_1 = M_{P+1}$) we have the additional Poisson relations

$$\{(TM^{-1})_1\,,\, M_2\} \;=\; -(TM^{-1})_1\, s_{12}^{+}\, M_2 \;+\; M_2\, s_{12}^{-}\,(TM^{-1})_1\;,$$

and

$$\{M_1\,,\, M_2\} \;=\; r_{12}^{+} M_1 M_2 \;-\; M_2 M_1 r_{12}^{-}\;.$$

which guarantee the symplecticity (Poisson) of the map.

Spectrum

More generally, the lattice KdV monodromy matrix is the ordered product of matrices $V_j(\lambda)$ along an initial-value staircase (with $\lambda := \lambda_{2j}$, $\lambda_{2j+1} = \lambda - a$):

$$T(\lambda) = \overleftarrow{\prod_{j=1}^{P}} V_j(\lambda) =: \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} , \quad V_j = \begin{pmatrix} v_j & 1 \\ \lambda_j & 0 \end{pmatrix} ,$$

where the period P can be either *even* ($P = 2N$) or *odd* ($P = 2N+1$), leading to the following gradings:

$$\text{Odd Case}: \; T(\lambda) = \begin{pmatrix} \lambda^g A_g + \cdots + A_0 & \lambda^g + \lambda^{g-1} B_{g-1} + \cdots + B_0 \\ \lambda\left(\lambda^g + \lambda^{g-1} C_{g-1} + \cdots + C_0\right) & \lambda\left(\lambda^{g-1} D_{g-1} + \cdots + D_0\right) \end{pmatrix}$$

$$\text{Even Case}: \; T(\lambda) = \begin{pmatrix} \lambda^{g+1} + \lambda^g A_g + \cdots + A_0 & \lambda^g B_g + \cdots + B_0 \\ \lambda\left(\lambda^g C_g + \cdots + C_0\right) & \lambda\left(\lambda^g + \lambda^{g-1} D_{g-1} + \cdots + D_0\right) \end{pmatrix}$$

where the *genus* $g = N$ and $g = N - 1$ respectively.

We need two types of spectral data:

i) the *main spectrum* given by the branch points of the spectral curve, i.e. the (assumed simple) roots $\lambda_1^0, \lambda_2^0, \ldots, \lambda_{2g+1}^0$ of the discriminant of the spectral curve $\det(\eta\mathbf{1} - T(\lambda)) = 0$:

$$R(\lambda) = \tfrac{1}{4}\left(tr\; T(\lambda)\right)^2 - \det T(\lambda) = B_g C_g \prod_{j=1}^{2g+1} (\lambda - \lambda_j^0) \, .$$

ii) the *auxiliary spectrum* corresponding to fixed boundary conditions, namely the roots of the (12)-entry of the monodromy matrix:

$$B(\lambda) = 0 \quad \Rightarrow \quad \lambda = \mu_1, \ldots, \mu_g \, .$$

Classical discrete Dubrovin equations

An explicit first-order discrete analogue of the Dubrovin eqs. was derived in [FWN,2000, loc. cit.]. Under the discrete-time evolution

$$T(\lambda) \quad\rightarrow\quad \widetilde{T}(\lambda) = M(\lambda)\, T(\lambda)\, M(\lambda)^{-1}\ , \quad \text{where} \quad M(\lambda) = \begin{pmatrix} w & 1 \\ \lambda & 0 \end{pmatrix}\ ,$$

the dynamics of the auxiliary spectrum is given by:

$$\boxed{\widetilde{\mathscr{M}}^{-1}\left(\sqrt{R(\widetilde{\boldsymbol{\mu}})} - \tfrac{1}{2} I_0 \mathbf{e}\right) + \mathscr{M}^{-1}\left(\sqrt{R(\boldsymbol{\mu})} + \tfrac{1}{2} I_0 \mathbf{e}\right) = 0}$$

where $\mathbf{e} = (1, 1, \ldots, 1)^t$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_g)^t$.
I_0 denotes a principal invariant of the underlying discrete system which is in terms of a discrete-time dynamics

$$(\mu_i, \eta_i) \ \rightarrow\ (\widetilde{\mu}_i, \widetilde{\eta}_i)\ , \quad (i = 1, \ldots, g)$$

with (μ_i, η_i) on the hyperelliptic curve.

$\mathscr{M}$ denotes the van der Monde matrix: $\quad \mathscr{M} = \begin{pmatrix} \mu_1 & \cdots & \mu_1^g \\ \vdots & \vdots & \\ \mu_1 & \cdots & \mu_g^g \end{pmatrix}$

Remarks:
1) The above explicit form of the dDubr eqs. contrasts an implicit form given[7]. A 2nd order form predicted in [Dubrovin et al., 1976] was given in [FWN & Enolskii, 1996]).
2) A more complicated version for 'close to the identity' maps was given in the context of geometric integration[8]

[7] S.J. Al'ber, *On Finite-Zone Solutions of Relativistic Toda Lattices*, Lett. Math. Phys. **17** (1989) 149–155.

[8] S. Al-Sallami, J. Niesen & FWN, *Closed-form modified Hamiltonians for integrable numerical integration schemes*, Nonlinearity **31** (2018) 5110.

Derivation

Writing the discrete-time evolution for the entries of the monodromy matrix $T(\lambda)$ in the form

$$\begin{aligned} \widetilde{A}(\lambda) &= wB(\lambda) + D(\lambda) \quad , \quad & \widetilde{C}(\lambda) &= \lambda B(\lambda) \\ \lambda \widetilde{B}(\lambda) &= C(\lambda) + w(A(\lambda) - D(\lambda)) - w^2 B(\lambda) \quad , \quad & \widetilde{D}(\lambda) &= A(\lambda) - wB(\lambda) \ . \end{aligned}$$

Expanding the first eq. in powers of λ we are led to the following set of equations:

$$\widetilde{A}_0 = I_0 = wB_0 \ \ , \ \ \widetilde{A}_j = wB_j + D_{j-1} \ \ , \ \ j = 1, \dots, g \ .$$

Note that we have $\widetilde{B}_g = C_g, \widetilde{C}_g = B_g$, and hence these top coefficient can be taken constant.

The entry $B(\lambda)$ has the following factorisation $B(\lambda) = B_g \prod_{j=1}^{g} (\lambda - \mu_j)$ leading to the expressions for $B_0/B_g, \dots, B_{g-1}/B_g$ as elementary symmetric functions of the zeroes $\mu_1, \dots, \mu_g$. The discrete Dubrovin equations arise now from the fact that

$$R(\lambda) = \tfrac{1}{4}(A(\lambda) - D(\lambda))^2 + B(\lambda)\, C(\lambda) \quad \Rightarrow \quad \tfrac{1}{2}(A - D)(\mu) = \kappa \sqrt{R(\mu)} \ ,$$

(with $\kappa = \pm$ the sign determining the sheet of the Riemann surface) which, with invariants $\mathrm{tr}(T(\lambda)) = (A + D)(\lambda) = I_0 + \sum_{j=1}^{g} \lambda^j I_j$, gives rise to the system

$$\begin{pmatrix} \frac{1}{2} I_1 - D_0 \\ \vdots \\ \frac{1}{2} I_g - D_{g-1} \end{pmatrix} = \begin{pmatrix} \mu_1 & \cdots & \mu_1^g \\ \vdots & & \vdots \\ \mu_g & \cdots & \mu_g^g \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sqrt{R(\mu_1)} - \frac{1}{2} I_0 \\ \vdots \\ \sqrt{R(\mu_g)} - \frac{1}{2} I_0 \end{pmatrix}$$

Derivation Ctd.

Furthermore, we have $wB(\lambda) = (\widetilde{A} + \widetilde{D})(\lambda) - D(\lambda) - \widetilde{D}(\lambda)$, and hence $I_j - D_{j-1} - \widetilde{D}_{j-1} = wB_j$, $(j = 1, \cdots, g)$, and $I_0 = wB_0$.
Hence, we need the expressions for B_j/B_0, $(j = 0, \cdots, g-1)$, in terms of symmetric functions of the μ_i. In fact, we can use the relation $B_0\mathbf{e} + \mathscr{M}\mathbf{b} = 0$, where $\mathbf{b} := (B_1, \cdots, B_g)^t$.
Note that the terms with I_0 in the dDubr eqs. can be expressed in terms of symmetric functions via

$$\mathscr{M}^{-1}\cdot\mathbf{e} = -\mathbf{S}(-\mu_1^{-1}, \cdots, -\mu_g^{-1}) , \quad \text{where} \quad S_k(x_1, \cdots, x_g) \equiv \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} .$$

Reconstruction formula: Follows from the relation $w = [(\widetilde{A} - \widetilde{D}) + (A - D)]/2B$, together with the relation $\frac{1}{2}(A - D) = \sqrt{R - BC}$. Then from the asymptotic limit $\lambda \to \infty$ we can find the reconstruction formula for the quantity w, namely

$$w = \kappa\left(\sqrt{\mathscr{A}} - \sqrt{\widetilde{\mathscr{A}}}\right) , \quad \text{where} \quad \mathscr{A} = \sum_{j=1}^{g}(\mu_j + \underset{\sim}{\mu}_j) - \sum_{j=1}^{2g+1} \lambda_j^0 .$$

Second-order discrete Dubrovin equations: This is the form that is closer to the one found in the literature [Matveev et al. 1976, Nijhoff & Enolskii, 1996] namely:

$$\widetilde{B}(\mu_i) - \underset{\sim}{B}(\mu_i) = 2\frac{w}{\mu_i}\sqrt{R(\mu_i)} .$$

Examples:

$\boxed{g=1}$

$$\frac{1}{2} l_0 \left(\frac{1}{\widetilde{\mu}} - \frac{1}{\mu} \right) = \frac{\sqrt{R(\mu)}}{\mu} + \frac{\sqrt{R(\widetilde{\mu})}}{\widetilde{\mu}}$$

which can be parametrised in terms of the Weierstrass $\wp$-function:

$$\mu = \wp(\delta) - \wp(t)\,, \quad \widetilde{\mu} = \wp(\delta) - \wp(t+\delta)\,, \quad \text{setting} \quad -\tfrac{1}{2} l_0 = \wp'(\delta)\,, \quad \sqrt{R(\mu)} = \wp'(t)\,.$$

$\boxed{g=2}$

$$\begin{aligned}
\tfrac{1}{2} l_0 \left(\frac{1}{\mu_1\mu_2} - \frac{1}{\widetilde{\mu}_1\widetilde{\mu}_2} \right) &= \frac{\frac{1}{\mu_1}\sqrt{R(\mu_1)} - \frac{1}{\mu_2}\sqrt{R(\mu_2)}}{\mu_1 - \mu_2} + \frac{\frac{1}{\widetilde{\mu}_1}\sqrt{R(\widetilde{\mu}_1)} - \frac{1}{\widetilde{\mu}_2}\sqrt{R(\widetilde{\mu}_2)}}{\widetilde{\mu}_1 - \widetilde{\mu}_2}\,, \\
\tfrac{1}{2} l_0 \left(\frac{\mu_1+\mu_2}{\mu_1\mu_2} - \frac{\widetilde{\mu}_1+\widetilde{\mu}_2}{\widetilde{\mu}_1\widetilde{\mu}_2} \right) &= \frac{\frac{\mu_2}{\mu_1}\sqrt{R(\mu_1)} - \frac{\mu_1}{\mu_2}\sqrt{R(\mu_2)}}{\mu_1 - \mu_2} + \frac{\frac{\widetilde{\mu}_2}{\widetilde{\mu}_1}\sqrt{R(\widetilde{\mu}_1)} - \frac{\widetilde{\mu}_1}{\widetilde{\mu}_2}\sqrt{R(\widetilde{\mu}_2)}}{\widetilde{\mu}_1 - \widetilde{\mu}_2}\,,
\end{aligned}$$

which can be parametrised in terms of the Kleinian ($g=2$ analogues of the Weierstrass $\wp$) functions.
These equations arise from the prototypical model of mappings of KdV type (i.e. of finite-dimensional reduction of KdV lattice).

Question: Is there a discrete analogue of the Abel map and the Jacobi inversion problem associated with these discrete equations?

Interpolating flow

The Poisson brackets between the entries of the monodromy matrix arise from the classical r-matrix structure. In particular they lead to:

$$\begin{aligned}\{A(\lambda_1), B(\lambda_2)\} &= \frac{A(\lambda_2)B(\lambda_1) - A(\lambda_1)B(\lambda_2)}{\lambda_1 - \lambda_2} + \frac{1}{\lambda_2} B(\lambda_1)D(\lambda_2) \ , \\ \{B(\lambda_1), D(\lambda_2)\} &= \frac{D(\lambda_2)B(\lambda_1) - (\lambda_2/\lambda_1)D(\lambda_1)B(\lambda_2)}{\lambda_1 - \lambda_2} \ ,\end{aligned}$$

which are preserved under the discrete KdV map. Note also that the coefficient B_g of $B(\lambda)$ is a Casimir. Thus, we find from

$$\{\mathrm{tr}(T(\lambda_1)), B(\lambda_2)\} = \frac{(A-D)(\lambda_2)\, B(\lambda_1) - (A-D)(\lambda_1)\, B(\lambda_2)}{\lambda_1 - \lambda_2} \ ,$$

from which we can take the limit $\lambda_2 \to \lambda_1$, and inserting the expansion for $B(\lambda)$ we find for the auxiliary spectrum the Poison relations

$$\{\mu_i, (A+D)(\mu_i)\} = (A-D)(\mu_i) \ , \quad \{(A+D)(\mu_i), \mu_j\} = 0 \quad \text{for} \quad j \neq i \ ,$$

while from the fundamental Poisson relations such as $\{B(\lambda_1), B(\lambda_2)\} = 0$ we can infer the canonical relations

$$\{\mu_i, \mu_j\} = \{\eta_i, \eta_j\} = 0 \ , \quad \{\eta_i, \mu_j\} = \eta_i \delta_{i,j} \ .$$

Furthermore, we find from the Poisson structure the generating Hamiltonian flow

$$\dot{\mu}_i := \{tr(T(\lambda))\, , \, \mu_i\} = \frac{(A-D)(\mu_i)\, B(\lambda)}{(\mu_i - \lambda)B'(\mu_i)} = \frac{2\sqrt{R(\mu_i)}\, B(\lambda)}{(\mu_i - \lambda)B_g \prod_{j\neq i}(\mu_i - \mu_j)} \ ,$$

which are the continuous Dubrovin equations for an interpolating flow t.

Integration and Jacobi inversion problem

Using the Lagrange interpolation formula:

$$\frac{\mu^m}{P(\mu)} + \sum_{j=1}^{n} \frac{\mu_j^m}{P'(\mu_j)(\mu_j - \mu)} = \left\{ \begin{array}{lll} 0 & , & 0 \leq m \leq n-1 \\ 1 & , & m = n \end{array} \right. \quad ,$$

in which $P(\mu) = \prod_{j=1}^{n}(\mu - \mu_j)$, we can integrate the Dubrovin equations, leading to the *Abel map*

$$\sum_{i=1}^{g} \int_{\mu_i(t_0)}^{\mu_i(t)} \frac{\mu_i^k d\mu_i}{\sqrt{R(\mu_i)}} = -\lambda^k (t - t_0) \quad , \quad (k = 0, \dots, g-1) \; ,$$

from the hyperelliptic Riemann surface of genus $g = N - 1$ (even case) or $g = N$ (odd case)

$$\Gamma : \quad \eta^2 - (A + D)(\lambda)\eta + \left\{ \begin{array}{c} \lambda^N (\lambda - a)^N \\ \lambda^{N+1} (\lambda - a)^N \end{array} \right. = 0,$$

to the Jacobian $\mathbb{C}^g/\Lambda$. The corresponding *Jacobi inversion problem* can be solved i.t.o. Riemann theta-functions using standard techniques of algebraic geometry. Alternatively, the solution of the KdV map can be found i.t.o. corresponding *Kleinian* functions[9] (i.e. higher genus Weierstrass functions).

Question: While we can solve the map using the interpolating flow, can we integrate the discrete Dubrovin equations directly?

[9] The integration of classical KdV mappings was first done in:
FWN & V. Enolskii, *Integrable Mappings of KdV Type and Hyperelliptic Addition Theorems*, in: Symmetries and Integrability of Difference Equations, P.A. Clarkson and F.W. Nijhoff, eds., LMS Lect. Notes series **255** (1999) 64–78.

Quantum Mappings of KdV Type (even case):

These are defined through a set of Hermitian quantum operators v_j ($j = 1, \ldots, 2N$) acting on some Hilbert space $\mathscr{H}$ on which they are assumed invertible (we want operators v_j^{-1} to exist).
Commutation relations:

$$\left[v_j \, , \, v_{j'}\right] \;=\; h\left(\delta_{j,j'+1} - \delta_{j+1,j'}\right) \quad , \quad (h = -i\hbar) \, ,$$

imposing periodicity condition $v_{i+N} = v_i$, $\delta_{i,j}$ is understood *modulo* N in each of the labels.
Discrete-time quantum map:

$$\widetilde{v}_{2j-1} = v_{2j} \qquad , \qquad \widetilde{v}_{2j} = v_{2j+1} + a{v_{2j}}^{-1} \;-\; a{v_{2j+2}}^{-1} \qquad (j = 1, \cdots, N),$$

where $a \in \mathbb{R}$ is a coupling constant, preserves the commutation relations.
The sums

$$\sum_{j=1}^{N} v_{2j} \;=\; \sum_{j=1}^{N} v_{2j-1} \;=:\; \nu \; ,$$

are Casimirs of the quantum algebra generated by the v_j, with ν treated as a free parameter of the (reduced) model.
The quantum mapping allows a Lax representation:

$$\widetilde{L}_j M_j = M_{j+1} L_j \; ,$$

taking into account the natural operator ordering along the lattice labeled by index j.

Here we have the Lax matrices (restricting ourselves to the even case):

$$L_j = V_{2j} V_{2j-1} \quad , \quad V_i = \begin{pmatrix} v_i & 1 \\ \lambda_i & 0 \end{pmatrix}$$

with $\lambda_{2j} = \lambda$, $\lambda_{2j+1} = \lambda - a$, and

$$M_n = \begin{pmatrix} w_n & 1 \\ \lambda & 0 \end{pmatrix} \quad , \quad w_j = v_{2j-1} - \frac{a}{v_{2j}} \ .$$

The *monodromy matrix*, $T(\lambda)$, is given by (with genus $g = N - 1$, period $P = 2N$)

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} := \overset{\longleftarrow}{\prod_{n=1}^{N}} L_n(\lambda) \ ,$$

which has the natural grading $T(\lambda)$ w.r.t. the spectral parameter λ, namely

$$T(\lambda) = \begin{pmatrix} \lambda^N + \lambda^{N-1} A_{N-1} + \ldots + A_0 & \lambda^{N-1} B_{N-1} + \lambda^{N-2} B_{N-2} + \ldots + B_0 \\ \lambda^N C_N + \lambda^{N-1} C_{N-1} + \ldots + \lambda C_1 & \lambda^N + \lambda^{N-1} D_{N-1} + \ldots + \lambda D_1 \end{pmatrix} .$$

From it we have a complete set of N *commuting invariants of the quantum mapping* obtained from:

$$\tau(\lambda) = A(\lambda) + \left(1 + \frac{h}{\lambda}\right) D(\lambda) = I_0 + I_1 \lambda + \cdots + I_{N-1} \lambda^{N-1} + 2\lambda^N \ ,$$

noting that

$$\widetilde{T}(\lambda) = M(\lambda) T(\lambda) M(\lambda)^{-1} \quad , \quad M(\lambda) = M_1(\lambda) \ .$$

Non-ultralocal Yang-Baxter (YB) structure:

This was established in [FWN & H. Capel, 1992] following similar structures developed by O. Babelon & L. Bonora (1991).
The non-ultralocal structure is given by the commutation relations:

$$\begin{aligned} R_{12}^{+}\, L_{n,1}\, L_{n,2} &= L_{n,2}\, L_{n,1}\, R_{12}^{-} \;, \\ L_{n+1,1} S_{12}^{+}\, L_{n,2} &= L_{n,2}\, L_{n+1,1} \;, \\ L_{n,1}\, L_{m,2} &= L_{m,2}\, L_{n,1} \qquad \mid n-m \mid \geq 2 \;, \end{aligned}$$

where $L_n(\lambda)$ is the L operator at the nth site.
$L_{n,j}$ denotes the tensor product:

$$L_{n,j} := \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes \underbrace{L_n(\lambda_j)}_{\text{jth place}} \otimes \ldots \otimes \mathbf{1}.$$

The YB matrices $R_{jk}^{\pm} := R_{jk}^{\pm}(\lambda_j, \lambda_k)$ act nontrivially only on the jth and kth factors of the tensor product. Similarly, $S_{j,k}^{\pm} := S_{j,k}^{\pm}(\lambda_j, \lambda_k)$ is an auxiliary YB matrix.
The consistency of the commutation relations require YB triangle relations:

$$\begin{aligned} R_{12}^{\pm}\, R_{13}^{\pm}\, R_{23}^{\pm} &= R_{23}^{\pm}\, R_{13}^{\pm}\, R_{12}^{\pm} \;, \\ R_{23}^{\pm}\, S_{12}^{\pm}\, S_{13}^{\pm} &= S_{13}^{\pm}\, S_{12}^{\pm}\, R_{23}^{\pm} \;, \end{aligned}$$

and there the condition ensuring that the local commutation relations can be "integrated":

$$R_{12}^{\pm}\, S_{12}^{\pm} \;=\; S_{12}^{\mp}\, R_{12}^{\mp} \;.$$

Extended Non-ultralocal Yang-Baxter (YB) structure:

The new element entering in the discrete-time case is *that including the the time-part of the Lax pair in the YB structure becomes essential !*
This is because there are no (known) effective ways of reconstructing the discrete-time matrix $M(\lambda)$ from the spatial part $L(\lambda)$, as is the case in the continous-time situation.
Thus, the extended YB structure was proposed [FWN, HW Capel, 1992] to include commutation relations for $M_n(\lambda)$, namely:

$$
\begin{aligned}
R_{12}^{+}\, M_{n,1}\, M_{n,2} &= M_{n,2}\, M_{n,1}\, R_{12}^{-} \ , \\
M_{n+1,1} S_{12}^{+}\, L_{n,2} &= L_{n,2}\, M_{n+1,1} \ , \\
M_{n,1}\, M_{m,2} &= M_{m,2}\, M_{n,1} \qquad \mid n-m \mid \geq 2 \ ,
\end{aligned}
$$

in addition to relations involving time-updates:

$$
\begin{aligned}
\widetilde{L}_{n,2} S_{12}^{-}\, M_{n,1} &= M_{n,1}\, \widetilde{L}_{n,2} \ , \\
\widetilde{M}_{n,1} S_{12}^{+}\, M_{n,2} &= M_{n,2}\, \widetilde{M}_{n,1} \ ,
\end{aligned}
$$

and all other commutation relations trivial (including between Lax matrices separated by higher time-updates).
Statement: The above set of commutation relations ensure that the discrete-time map $L_n \to \widetilde{L}_n = M_{n+1} L_n M_n^{-1}$ preserves the non-ultralocal commutation structure.

Example: The quantum mappings of KdV type provide an example of this structure with the following choice of YB matrices:

$$
\begin{aligned}
R^+_{12} &= R^-_{12} - S^+_{12} + S^-_{12} \\
R^-_{12} &= \mathbf{1} \otimes \mathbf{1} \; + \; h\,\frac{P_{12}}{\lambda_1 \, - \, \lambda_2} \\
S^+_{12} &= \mathbf{1} \otimes \mathbf{1} \; - \, \frac{h}{\lambda_2} F \otimes E \qquad , \qquad S^-_{12} = S^+_{21} \; ,
\end{aligned}
$$

where the permutation operator P_{12} and the matrices E and F are given by

$$
P_{12} \; = \; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad E \; = \; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F \; = \; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; .
$$

Remark: It was shown[10] that this example can be generalised to the entire class of lattice systems of "Gel'fand-Dikii type", i.e. systems of higher rank (including the lattice Boussinesq family).

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[10] FWN & H.W. Capel, *Integrability and fusion algebra for quantum mappings*, J. Phys. A: Math. Gen. **26** (1993) 6385–6407.

The quantum algebra

Integrating the non-ultralocal YB structure to the one of the monodromy matrix $T(\lambda)$ we get the relation

$$\boxed{\mathrm{R}_{12}^{+} T_1 S_{12}^{+} T_2 = T_2 S_{12}^{-} T_1 R_{12}^{-}} \quad \text{with} \quad T_1 = T(\lambda_1) \otimes \mathbf{1} \ , \ \ T_2 = \mathbf{1} \otimes T(\lambda_2) \ ,$$

leading to the following commutation relations:

$$[A_1, A_2] = \frac{h}{\lambda_2} B_1 C_2 - \frac{h}{\lambda_1} B_2 C_1 \ , \quad [B_1, B_2] = [C_1, C_2] = [D_1, D_2] = 0 \ ,$$

$$[A_1, B_2] + [B_1, A_2] = \frac{h}{\lambda_2} B_1 D_2 - \frac{h}{\lambda_1} B_2 D_1 \ , \quad \hat{\lambda}_{12} B_1 A_2 = h B_2 A_1 + \lambda_{12} \left(A_2 B_1 - \frac{h}{\lambda_1} B_2 D_1 \right)$$

$$[A_1, C_2] + \frac{\lambda_2}{\lambda_1} [C_1, A_2] = \frac{h}{\lambda_1} (D_1 C_2 - D_2 C_1) \ , \quad h \frac{\lambda_1}{\lambda_2} A_1 C_2 + \lambda_{12} \left(C_1 A_2 - \frac{h}{\lambda_2} D_1 C_2 \right)$$

$$\lambda_{12} [A_1, D_2] = h C_2 B_1 - h \frac{\lambda_2}{\lambda_1} C_1 B_2 \ , \quad h \frac{\lambda_1}{\lambda_2} B_1 C_2 + \lambda_{12} D_1 A_2 = h B_2 C_1 + \lambda_{12} A_2 D_1 \ ,$$

$$\lambda_{12} [B_1, C_2] = h D_2 A_1 - h \frac{\lambda_2}{\lambda_1} D_1 A_2 - \lambda_{12} \frac{h}{\lambda_1} D_2 D_1 \ , \quad \lambda_{12} (D_1 A_2 - A_2 D_1) = h B_2 C_1 - h \frac{\lambda_1}{\lambda_2} B_1 C_2$$

$$[B_1, D_2] = \frac{\lambda_2}{\lambda_1} (D_2 B_1 - D_1 B_2) \ , \quad h \frac{\lambda_1}{\lambda_2} B_1 D_2 + \lambda_{12} D_1 B_2 = \hat{\lambda}_{12} B_2 D_1 \ ,$$

$$\hat{\lambda}_{12} C_1 D_2 = \lambda_{12} D_2 C_1 + h C_2 D_1 \ , \quad [C_1, D_2] = [C_2, D_1] \ .$$

where $A_i = A(\lambda_i)$, $(i = 1, 2)$, etc. , and $\lambda_{12} = \lambda_1 - \lambda_2$, $\hat{\lambda}_{12} = \lambda_{12} + h$.

Quantum Invariants

On the basis of the YB structure quantum commuting invariants of the discrete-tme map can be constructed as follows.

Denoting by $M(\lambda) = M_1(\lambda)$ the discrete matrix at the initial point of the chain and by $M_j := \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes \underbrace{M(\lambda_j)}_{\text{jth place}} \otimes \ldots \otimes \mathbf{1}$ its entry in tensorial products (and similar for $T(\lambda)$), we find the "integrated" extended YB structure:

$$\left.\begin{array}{rcl} R^+_{12}\, T_1\, S^+_{12}\, T_2 & = & T_2\, S^-_{12}\, T_1\, R^-_{12}\;, \\ R^+_{12}\, M_1 M_2 & = & M_2 M_1 R^-_{12}\;, \\ T_1\, M_1^{-1}\, S^+_{12}\, M_2 & = & M_2\, S^-_{12}\, T_1\, M_1^{-1}\;. \end{array}\right\} \quad \Rightarrow \quad R^+_{12}\, \widetilde{T}_1\, S^+_{12}\, \widetilde{T}_2 = \widetilde{T}_2\, S^-_{12}\, \widetilde{T}_1\, R^-_{12}\;.$$

There are two distinct requirements:

1. quantum invariants should be invariant under the map;
2. quantum invariants should commute.

Satisfying both conditions is guaranteed if we implement a nontrivial extension of quantum traces using Sklyanin boundary K matrices[11] leading to:

$\boxed{\tau(\lambda) \;=\; \mathrm{tr}\,(K(\lambda)T(\lambda))}$, in spite of the fact that we work with periodic boundary conditions!

Explanation: the K matrices are needed to account for the non-periodicity in time!

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[11] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen. **21** (1988) 2375–2389.

Explicit form for the K matrix

The boundary matrix $K(\lambda)$ must obey the *reflection equation*:

$$K_1 \, {}^{t_1}\!\Big(({}^{t_1}S_{12}^-)^{-1}\Big) \, K_2 R_{12}^+ = R_{12}^- K_2 \, {}^{t_2}\big(({}^{t_2}S_{12}^+)^{-1}\big) \, K_1 \ ,$$

which surprisingly turns out to have an explicit solution:

$$K_2 \ = \ \mathrm{tr}_1 \left\{ P_{12} \, {}^{t_1}\left[(\, {}^{t_1}S_{12}^+)^{-1} \right] \right\} \ .$$

(subscripts 1,2 referring to entries in a tensorial product), and left superscript ^{t_1} denoting matrix transposition in the first factor of the matricial tensor product. Then the corresponding $\tau(\lambda)$ will obey both conditions, 1) & 2), for generating quantum invariants of the quantum map.
From the explicit form of the KdV S_{12}^+ matrix we thus find:
$K(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + h/\lambda \end{pmatrix}$, giving a quantum correction to the classical invariants.

"Spatial" evolution
A "spatial" evolution along the chain can be introduced by cyclic permutation along the periodic staircase defining $T(\lambda)$. Thus, in addition to the "temporal" evolution we can consider the map:

$$\widehat{T}(\lambda) = L_1(\lambda) T(\lambda) L_1(\lambda)^{-1} \ ,$$

which shares many of the properties with the time-map. Indeed:

$$T_1 \, L_{1,1}^{-1} \, S_{12}^+ \, L_{1,2} = L_{1,2} \, S_{12}^- \, T_1 \, L_{1,1}^{-1} \ .$$

Thus, spatial and temporal evolution can be treated on almost the same footing.

Quantum Determinants

We further need in the construction the notion of quantum determinants introduced (in the ultralocal case) by Reshetikhin, Kulish & Sklyanin (1981).
Quantum analogies of determinants can be considered under the assumption that the R-matrix proportional to a rank-1 projector for a particular relative value of the spectral parameters λ_1, λ_2.
Example: for the KdV quantum structure we can set

$$\lambda_2 = \lambda_1 + h \quad \Rightarrow \quad R^-_{12} = \mathbf{1} - P_{12} \quad \text{a rank 1 projector!}$$

From

$$R^-_{12}\, S^-_{12}\, T_1\, S^+_{12}\, T_2 = S^+_{12}\, T_2\, S^-_{12}\, T_1\, R^-_{12}\ ,$$

we can then identify:

$$\boxed{R^-_{12}\, S^-_{12}\, T_1\, S^+_{12}\, T_2 := R^-_{12}\, \Delta}$$

where $\Delta =: \mathrm{Qet}(T)$ is (scalar) operator which is the (global) quantum determinant.
In a similar fashion **quantum minors** can be constructed (for the higher rank case) using a "fusion procedure".
In a similar fashion we can consider *local* quantum determinants denoted by $\mathrm{Qet}(L_n)$, which are defined through:

$$R^-_{12}\mathrm{Qet}(L_n) = R^-_{12} S^-_{12}\, L_{n,1}\, L_{n,2} = S^+_{12} L_{n,2}\, L_{n,1}\, R^-_{12}\ .$$

(where again $\lambda_2 = \lambda_1 + h$).

Some important properties:
Using the extended YB structure the following can be established:

1. Δ is in the centre of the quantum algebra generated by the entries of the monodromy matrix: $[\Delta(\lambda), T(\mu)] = 0$ when the R-matrix is of "Yangian type" (as in the KdV case).
2. Δ is covariant under the quantum map, i.e.

$$\widetilde{\Delta} = \mathrm{Qet}(M)\,\Delta\,\mathrm{Qet}(M^{-1})$$

which implies that it is invariant, $\widetilde{\Delta} = \Delta$, in the Yangian type case.
3. For the monodromy matrix given by the operator-ordered product $T = \overleftarrow{\prod_j} L_j$ the quantum determinant factorises:

$$\Delta = \overleftarrow{\prod_{n=1}^{P}} \mathrm{Qet}(L_n).$$

In the case of the KdV quantum structure, by the definition we have

$$\Delta(\lambda) = \frac{\lambda_+}{\lambda_-}\left[A(\lambda_+)D(\lambda_-) - B(\lambda_+)C(\lambda_-)\right] \quad , \quad \lambda_\pm = \lambda \pm \frac{h}{2} \ .$$

Furthermore, prop. # 3 and the form of the Lax matrices L_j lead to:

$$\mathrm{Qet}(L_n) = \lambda_+(\lambda_+ - a) \quad \Rightarrow \quad \Delta(\lambda) = \lambda_+^N(\lambda_+ - a)^N \ .$$

Quantum Separated Variables

Following [Sklyanin,1992] the operator zeros of the B-entry of the monodromy matrix provide separated canonical variables:

$$B(\lambda) \ = \ B_{N-1} \prod_{n=1}^{N-1} (\lambda - x_n) \quad , \quad B_{N-1} \ \text{Casimir} \ \ \text{operator} \ .$$

The YB relations yields: $[\, B(\lambda) \, , \, B(\mu) \,] = 0$, from which we can infer that the roots x_n mutually commute[12].

The same analysis holds true for the non-ultralocal as for the ultralocal algebra, with only minor modifications of the algebraic relations. Introducing the conjugate operators to the x_n by:

$$X_n^- = \Big|_{\lambda = x_n} A(\lambda) \ , \qquad X_n^+ = \Big|_{\lambda = x_n} \left(1 + \frac{h}{\lambda}\right) D(\lambda)$$

with operator substitution from the left, i.e.

$$\begin{aligned} X_n^- \quad &= \quad A_0 + x_n A_1 + \cdots + x_n^N \\ X_n^+ \quad &= \quad hD_1 + x_n(D_1 + hD_2) + \cdots + x_n^N \ . \end{aligned}$$

With these definitions the following algebraic relations can be proven:

$$[x_m \, , \, x_n] = [X_m^{\pm} \, , \, X_n^{\pm}] = [X_m^- \, , \, X_n^+] = 0 \ , \quad (m \neq n) \ ,$$

$$X_m^{\pm} x_n = (x_n \pm h\, \delta_{mn}) X_m^{\pm} \ , \quad X_n^{\pm} X_n^{\mp} = \Delta \left(x_n \pm \frac{h}{2} \right) \ ,$$

(for $m, n = 1, \ldots, N-1$), where for $\Delta(\lambda)$ we have the explicit expression given above.
From the expression for $\tau(\lambda)$ with left substitution, we also have:

$$\tau(x_n) = X_n^+ + X_n^- ,$$

Quantum discrete Dubrovin equations (temporal part)

The following set of relations follow from the commutation relations:

$$[\,A - Bw\ ,\ w] = -\frac{h}{\lambda}(C - Dw) \quad , \quad [\,C - D\,w\,,\ w] = 0$$

$$[\,B\,,\ w\,] = \frac{h}{\lambda}(A - Bw - D) \quad , \quad [\,D\,,\ w\,] = \frac{h}{\lambda}(C - Dw)$$

where $w = w_1$, whilst the time evolution gives:

$$\widetilde{A} = wB + D \quad , \quad \lambda\widetilde{B} = wA - wBw + C - Dw \ ,$$
$$\widetilde{C} = \lambda B \quad , \quad \widetilde{D} = A - Bw$$

On the one hand, from the definition of $X_N^{\pm}$ we have

$$X_n^- - \frac{x_n - h}{x_n + h}X_n^+ = (A_0 + hD_1) + x_n(A_1 + hD_2 - D_1) + \cdots + x_n^{N-1}(A_{N-1} + h - D_{N-1}) \ ,$$

on the other hand from $\widetilde{A} - D = \frac{\lambda+h}{\lambda}Bw - \frac{h}{\lambda}(A - D)$ we have by left-substitution $\widetilde{A}(x_n) = X_n^+ - \frac{h}{x_n}X_n^-$, whilst $\widetilde{D}(x_n) = X_n^-$. Combining these we obtain:

$$X_n^+ - X_n^- = (\widetilde{A}_0 + h\widetilde{D}_1) + x_n(\widetilde{A}_1 - \widetilde{D}_1 + h\widetilde{D}_2) + \cdots + x_n^{N-1}(\widetilde{A}_{N-1} - \widetilde{D}_{N-1} + h) \ ,$$

(where $n = 1, \ldots, N-1$). Denoting by $\mathbf{X}^{\pm}$ the vector $(X_1^{\pm}, \ldots, X_{N-1}^{\pm})^t$, and introducing the VanderMonde matrix $\mathscr{M}$

$$\mathscr{M} = \begin{pmatrix} x_1 & x_1^2 & \cdots & x_1^{N-1} \\ x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & & & \vdots \\ x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1} \end{pmatrix} ,$$

and the diagonal matrix $\mathscr{D} = \mathrm{diag}(x_1, \ldots, x_n)$, the comparison yields:

Quantum discrete Dubrovin equations

$$\boxed{\widetilde{\mathscr{M}}^{-1}\Big(\widetilde{\mathbf{X}}^- + (h\mathbf{1}+\widetilde{\mathscr{D}})^{-1}(h\mathbf{1}-\widetilde{\mathscr{D}})\widetilde{\mathbf{X}}^+ - I_0\mathbf{e}\Big) = \mathscr{M}^{-1}\Big(\mathbf{X}^+ - \mathbf{X}^- - I_0\mathbf{e}\Big)}$$

where $I_0 = \tau(0) = A_0 + hD_1$ is invariant. This equation, together with

$$X_n^{\pm} X_n^{\mp} = \Delta(x_n \pm \frac{h}{2}) \quad , \quad \tau(x_n) = X_n^+ + X_n^-$$

constitutes the (temporal part of the) quantum discrete Dubrovin equations.

Reconstruction of potential w:
From the previous relations we have in particular

$$w = \left[\frac{\lambda+h}{2\lambda+h}\left(\widetilde{A}(\lambda) - \widetilde{D}(\lambda)\right) + \frac{\lambda}{2\lambda+h}\left(A(\lambda) - D(\lambda)\right)\right] B(\lambda)^{-1} ,$$

which, upon taking the limit $\lambda \to \infty$, yields

$$w = \frac{1}{2\nu}(\widetilde{A}_{N-1} - \widetilde{D}_{N-1} + A_{N-1} - D_{N-1}) ,$$

in which $\nu = B_{N-1} = B_g$ is a Casimir. Expressions for $A_{N-1} - D_{N-1}$ in terms of the $\{x_n, X_n^{\pm}\}$ can be obtained from pervious formulae yielding the following result:

$$w = \frac{1}{\nu}\Bigg[\nu^2 - Na + (N-1)h + (-1)^N \sum_{n=1}^{N-1}\Bigg(\prod_{\substack{i=1\\ i\neq n}}^{N-1} \frac{1}{x_i - x_n}\Bigg) \times \left(2x_n^{N-1} - \frac{1}{h+x_n}X_n^+ - \frac{1}{x_n}X_n^-\right)\Bigg].$$

Example: the $N = 2$ case (genus 1)

In the original variables $v_1, \ldots, v_4$, subject to the periodicity conditions $v_2 + v_4 = v_1 + v_3 = \nu$ (ν Casimir), quantum map reads:

$$\widetilde{v}_1 = v_2 \quad , \quad \widetilde{v}_2 = \nu - v_1 + a v_2^{-1} - a(\nu - v_2)^{-1} \ ,$$

subject to $[\, v_2 \, , \, v_1 \,] = h = -i\hbar$. The quantum invariant is an operator biquadratic:

$$I_0 = [(\nu - v_2)(\nu - v_1) - a]\, [v_2 v_1 - a] + h(\nu - v_1) v_2 \ .$$

In this case the separated variables are $x, X^{\pm}$ obeying the relations:

$$X^{\pm} x = (x \pm h) X^{\pm} \quad , \quad X^{\pm} X^{\mp} = \Delta(x \pm \frac{h}{2}) \ ,$$

where:

$$\Delta(\lambda) = (\lambda + \frac{h}{2})^2 (\lambda + \frac{h}{2} - a)^2 \quad , \quad \tau(x) = I_0 + x I_1 + 2x^2 = X^{+} + X_{-} \ ,$$

where $I_1 = \nu^2 + h$ is a Casimir. The quantum discrete Dubrovin eqs. read:

$$x^{-1}(X^{+} - X^{-} - I_0) = \widetilde{x}^{-1}(\widetilde{X}^{-} + (h - \widetilde{x})(h + \widetilde{x})^{-1} \widetilde{X}^{+} - I_0) \ ,$$

In this case only the temporal quantum discrete Dubrovin eq. is needed, which can be recast in the form:

$$\tilde{x} - \frac{1}{\tilde{x} + h} \widetilde{X}^{+} = x - \frac{1}{x} X^{-}.$$

together with the reconstruction formula for $w = \widetilde{v}_2 - a v_2^{-1}$:

$$w = \frac{1}{\nu} \left(\nu^2 - 2a + h + 2x - \frac{1}{x + h} X^{+} - \frac{1}{x} X^{-} \right) .$$

Furthermore,

$$\tilde{x} = \frac{1}{\nu} w \left(x - \frac{1}{x} X^- \right) .$$

It is easily seen that

$$\left(1 + \frac{h}{x}\right) \frac{1}{x+h} X^+ = -\frac{1}{x} X^- + \frac{1}{x} I_0 + I_1 + 2x .$$

Hence we may consider the evolution given by the quantum discrete Dubrovin equations to be that of x and $\frac{1}{x+h} X^+$ (or $\frac{1}{x} X^-$) together with the preservation of the invariant, $\widetilde{I_0} = I_0$, whereas $I_1 = \nu^2 - 2(a-h)$ in this case.

Final remarks:

1. We have shown that there exist a proper quantum and discrete analogue of the classical Dubrovin equations arising in the finite-gap integration theory. They are the discrete operator equations of the motion in terms of the separated variables of the system (Sklyanin variables).
2. The main question is to establish in what way they provide a possible route to "solving" the quantum discrete-time model, and to make sense of the quantum model defined through the operator equations of the motion.
3. Since these quantum models (in parallel to the classical case) exhibit *singularities*, one may rephrase this by asking **to what extent there exists a quantum (discrete) analogue of the Painlevé property?**
4. What does *'solving the quantum model'* mean in this context?
   - finding closed-form expressions for the quantum propagators?
   - setting up the Hilbert-space picture (i.e. representation theory) for these quantum models?