Solutions to Non-Commutative Integrable Systems

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A System with Pfaffian Solutions

Hirota and Ohta coupled soliton equations:

$$(4u_t - 6uu_x - u_{xxx})_x - 3u_{yy} + 24(v\bar{v})_{xx} = 0$$

$$2v_t + 3uv_x + v_{xxx} - 3\left(v_{xy} + v\int^x u_y\,dx\right) = 0$$

$$2\bar{v}_t + 3u\bar{v}_x + \bar{v}_{xxx} + 3\left(\bar{v}_{xy} + \bar{v}\int^x u_y\,dx\right) = 0$$

With the transformation

$$u = 2(log\tau)_{xx}, \qquad v = \frac{\sigma}{\tau}, \qquad \bar{v} = \frac{\bar{\sigma}}{\tau}$$

we can write the equations in bilinear form. The τ, σ and $\bar{\sigma}$ can be expressed as Pfaffians.

KP equation

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0.$$

The non-commutative KP equation:

$$(v_t + v_{xxx} + 3v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = 0$$

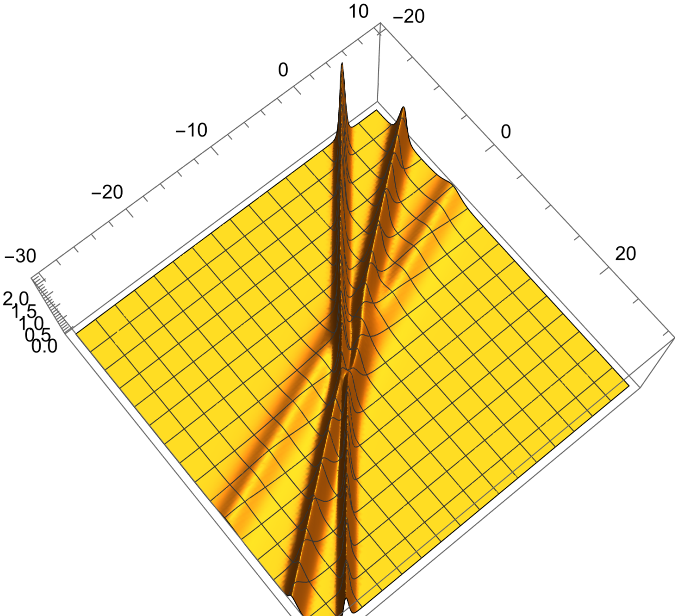
If the variables commute we can set $v_x = u$ and get back to the conventional KP equation.
Associated Lax pair for the ncKP equation is

$$L\phi = (\partial_x^2 + v_x - \partial_y)\phi = 0,$$
$$M\phi = (4\partial_x^3 + 6v_x\partial_x + 3v_{xx} + 3v_y - \partial_t)\phi = 0.$$

Consistency between the two operators gives the nc KP equation i.e.

$$[L, M] = 0$$

Solitons



Quasi-determinants (Etingof, Gelfand and Retakh)

For a block matrix

$$\begin{pmatrix} A & B \\ C & d \end{pmatrix}$$

where d is a single entry, A is a square matrix (say $n \times n$) and B, C are column and row vectors, one quasi-determinant is

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B.$$

Indeed

$$\left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix}^{-1} \right\}_{n+1,n+1} = (d - CA^{-1}B)^{-1}.$$

If the entries in our matrix commute, then

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = \frac{\begin{vmatrix} A & B \\ C & d \end{vmatrix}}{|A|} \qquad \text{(Cramer's rule?)}$$

Quasi-determinants

Example 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are all single entries.
one quasi-determinant is

$$\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b.$$

There are other quasi determinants obtained by expanding about different entries:

$$\begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a \qquad \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c$$

and $$\begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d$$

Quasi-determinants

Example 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are all single entries.
one quasi-determinant is

$$\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b \quad \text{(in commutative case } = \begin{vmatrix} a & b \\ c & d \end{vmatrix} / a)$$

There are other quasi determinants obtained by expanding about different entries:

$$\begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a \qquad \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c$$

and $$\begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d$$

Quasi Determinant identities

For Determinants we have Jacobi's identity:

$$\begin{vmatrix} A & B_1 & B_2 \\ C_1 & d_1 & d_2 \\ C_2 & d_3 & d_4 \end{vmatrix} |A| = \begin{vmatrix} \begin{vmatrix} A & B_1 \\ C_1 & d_1 \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & d_2 \end{vmatrix} \\ \\ \begin{vmatrix} A & B_1 \\ C_2 & d_3 \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_2 & d_4 \end{vmatrix} \end{vmatrix}$$

Quasi Determinant identities

Like normal determinants, quasideterminants obey identities. The analogue of the Jacobi identity for determinants is Sylvester's Identity

$$\begin{vmatrix} A & B_1 & B_2 \\ C_1 & d_1 & d_2 \\ C_2 & d_3 & \boxed{d_4} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} A & B_1 \\ C_1 & \boxed{d_1} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & \boxed{d_2} \end{vmatrix} \\ & \\ \begin{vmatrix} A & B_1 \\ C_2 & \boxed{d_3} \end{vmatrix} & \boxed{\begin{vmatrix} A & B_2 \\ C_2 & \boxed{d_4} \end{vmatrix}} \end{vmatrix}$$

A second Quasi Determinant identity

There are other quasideterminant identities as well, for example:

$$\begin{vmatrix} A & B_1 & B_2 & B_3 \\ C_1 & d_1 & d_2 & d_3 \\ C_2 & d_4 & d_5 & d_6 \\ C_3 & d_7 & d_8 & \boxed{d_9} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} A & B_1 \\ C_1 & \boxed{d_1} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & \boxed{d_2} \end{vmatrix} & \begin{vmatrix} A & B_3 \\ C_1 & \boxed{d_3} \end{vmatrix} \\ \begin{vmatrix} A & B_1 \\ C_2 & \boxed{d_4} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_2 & \boxed{d_5} \end{vmatrix} & \begin{vmatrix} A & B_3 \\ C_2 & \boxed{d_6} \end{vmatrix} \\ \begin{vmatrix} A & B_1 \\ C_3 & \boxed{d_7} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_3 & \boxed{d_8} \end{vmatrix} & \boxed{\begin{vmatrix} A & B_3 \\ C_3 & \boxed{d_9} \end{vmatrix}} \end{vmatrix}$$

This turns out to be useful for skew-symmetric cases.

Differentiation of a quasi-determinant

Consider the derivative

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B - C(A^{-1})'B - CA^{-1}B'$$
$$= d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B'$$

Suppose now that A is a skew symmetric, with entries

$$A_{ij} = \int^x (f_i g_j - g_i f_j)dx \qquad \text{then} \qquad \frac{\partial A_{ij}}{\partial x} = (f_i g_j - g_i f_j)$$

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \begin{vmatrix} A & B' \\ C & \boxed{0} \end{vmatrix}$$
$$+ \begin{vmatrix} A & f \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ g^T & \boxed{0} \end{vmatrix} - \begin{vmatrix} A & g \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ f^T & \boxed{0} \end{vmatrix}.$$

Pfaffians

Let A be a $2n \times 2n$ skewsymetric matrix then

$$\det(A) = pf(A)^2$$

$pf(A)$ is a polynomial in the matrix entries.
If A is $(2n+1) \times (2n+1)$ then the $\det(A) = 0$.
You can write a pfaffian as a triangular array, eg

$$\left| \begin{matrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{matrix} \right| = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

Shorthand notation:

$$(1,2,3,4) = (12)(34) - (13)(24) + (14)(23)$$

where $(ij) = a_{ij}$.

Inverse of a Skew Symmetric Matrix

Take entries $(ij) = -(ji)$,

$$\begin{pmatrix} 0 & (12) & (13) & (14) & (15) & (16) \\ (21) & 0 & (23) & (24) & (25) & (26) \\ (31) & (32) & 0 & (34) & (35) & (36) \\ (41) & (42) & (43) & 0 & (45) & (46) \\ (51) & (52) & (53) & (54) & 0 & (56) \\ (61) & (62) & (63) & (64) & (65) & 0 \end{pmatrix}^{-1}$$

$$= \frac{1}{(123456)} \begin{pmatrix} 0 & (3456) & -(2456) & (2356) & -(2346) & (3456) \\ . & 0 & (1456) & -(1356) & (1346) & -(1345) \\ . & . & 0 & (1256) & -(1246) & (1245) \\ . & . & . & 0 & (1236) & -(1235) \\ . & . & . & . & 0 & (1236) \\ . & . & . & . & . & 0 \end{pmatrix}$$

Quasi Pfaffians

We could attempt to look at the quasideterminant of a skew symmetric matrix, this doesn't work:

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{0} \end{vmatrix} \qquad \text{with} \quad a_{ji} = -a_{ij}$$

or

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & \boxed{0} \end{vmatrix}$$

Quasi Pfaffians

For the quasi pfaffian we will take entries $a_{ij} = -a_{ji}^T$,
(will also need $a_{ii} = -a_{ii}^T$ rather than $a_{ii} = 0$)
For example

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{1\beta} \\ a_{21} & 0 & a_{23} & a_{24} & a_{2\beta} \\ a_{31} & a_{32} & 0 & a_{34} & a_{3\beta} \\ a_{41} & a_{42} & a_{43} & 0 & a_{4\beta} \\ a_{\alpha 1} & a_{\alpha 2} & a_{\alpha 3} & a_{\alpha 3} & \boxed{a_{\alpha\beta}} \end{vmatrix}$$

$$= (1, 2, 3, 4, \boxed{\alpha, \beta}).$$

Quasi Pfaffians (commutative case)

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{1\beta} \\ a_{21} & 0 & a_{23} & a_{24} & a_{2\beta} \\ a_{31} & a_{32} & 0 & a_{34} & a_{3\beta} \\ a_{41} & a_{42} & a_{43} & 0 & a_{4\beta} \\ a_{\alpha 1} & a_{\alpha 2} & a_{\alpha 3} & a_{\alpha 4} & \boxed{a_{\alpha\beta}} \end{vmatrix}$$

$$= a_{\alpha\beta} - \begin{pmatrix} a_{\alpha 1} & a_{\alpha 2} & a_{\alpha 3} & a_{\alpha 4} \end{pmatrix} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}^{-1} \begin{pmatrix} a_{1\beta} \\ a_{2\beta} \\ a_{3\beta} \\ a_{4\beta} \end{pmatrix}$$

$$= \frac{(1\,2\,3\,4\,\alpha\,\beta)}{(1\,2\,3\,4)}.$$

Back to Hirota and Ohta:

Hirota and Ohta coupled soliton equations:

$$(4u_t - 6uu_x - u_{xxx})_x - 3u_{yy} + 24(v\bar{v})_{xx} = 0$$

$$2v_t + 3uv_x + v_{xxx} - 3\left(v_{xy} + v\int^x u_y\,dx\right) = 0$$

$$2\bar{v}_t + 3u\bar{v}_x + \bar{v}_{xxx} + 3\left(\bar{v}_{xy} + \bar{v}\int^x u_y\,dx\right) = 0$$

With the transformation

$$u = 2(log\tau)_{xx}, \qquad v = \frac{\sigma}{\tau}, \qquad \bar{v} = \frac{\bar{\sigma}}{\tau}.$$

$$\tau = (1\,,2\,\cdots\,,2N),$$

$$\sigma = (1\,,2\,\cdots\,,2N, c_1, c_0), \qquad \bar{\sigma} = (1\,,2\,\cdots\,,2N, d_0, d_1),$$

Pfaffian entries have a skew symmetric form:

$$(i\,,j) = \int^x (f_i g_j - g_i f_j)dx, \quad (i\,,c_n) = \frac{\partial^n}{\partial x^n} g_i, \quad (i\,,d_n) = \frac{\partial^n}{\partial x^n} f_i$$

Back to Hirota and Ohta:

Hirota and Ohta coupled soliton equations:

$$u = 2(log\tau)_{xx}, \qquad v = \frac{\sigma}{\tau}, \qquad \bar{v} = \frac{\bar{\sigma}}{\tau}.$$

$$\tau = (1\,, 2\,\cdots\,, 2N),$$

$$\sigma = (1\,, 2\,\cdots\,, 2N, c_1, c_0), \qquad \bar{\sigma} = (1\,, 2\,\cdots\,, 2N, d_0, d_1),$$

$$(i\,, j) = \int^x (f_i g_j - g_i f_j)dx, \quad (i\,, c_n) = \frac{\partial^n}{\partial x^n} g_i, \quad (i\,, d_n) = \frac{\partial^n}{\partial x^n} f_i$$

The f's and g's obey simple linear relationships ($x_2 = y, x_3 = t$)

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial^n f_i}{\partial x^n}, \qquad \frac{\partial g_i}{\partial x_n} = (-1)^{n-1} \frac{\partial^n g_i}{\partial x^n}$$

The key to showing these are solutions is by differentiating the functions τ, σ and $\bar{\sigma}$.

$$\partial_x \tau = \partial_x (1\,, 2\,\cdots\,, 2N) = (1\,, 2\,\cdots\,, 2N, c_0, d_0).$$

Hirota and Ohta equations

Rewrite the system using $u = w_x$ then

$$w = 2(log\tau)_x = 2\frac{\tau_x}{\tau}, \qquad v = \frac{\sigma}{\tau}, \qquad \bar{v} = \frac{\bar{\sigma}}{\tau}.$$

$$\tau = (1\,, 2\,\cdots\,, 2N), \qquad \tau_x = (1\,, 2\,\cdots\,, 2N, c_0, d_0)$$

$$\sigma = (1\,, 2\,\cdots\,, 2N, c_1, c_0), \qquad \bar{\sigma} = (1\,, 2\,\cdots\,, 2N, d_0, d_1),$$

we get

$$(4w_t - 3(w_x)^2 - w_{xxx})_x - 3w_{yy} + 24(v\bar{v})_x = 0$$

$$2v_t + 3w_x v_x + v_{xxx} + 3(v_{xy} + vw_y) = 0$$

$$2\bar{v}_t + 3w_x\bar{v}_x + \bar{v}_{xxx} - 3(\bar{v}_{xy} + \bar{v}w_y) = 0.$$

Derivatives of a quasi-Pfaffian

$$\frac{\partial}{\partial x}\left(1\cdot\cdot 2N\,\boxed{d_r, c_s}\,\right) = \left(1\cdot\cdot 2N\,\boxed{d_{r+1}, c_s}\,\right) + \left(1\cdot\cdot 2N\,\boxed{d_r, c_{s+1}}\,\right)$$
$$+\left(1\cdots 2N\,\boxed{d_r,\ c_0}\,\right)\left(1\cdots 2N\,\boxed{d_0,\ c_s}\,\right)$$
$$-\left(1\cdots 2N\,\boxed{d_r,\ d_0}\,\right)\left(1\cdots 2N\,\boxed{c_0,\ c_s}\,\right)$$

Compact notation

$$Q[d_r, c_s]_x = Q[d_{r+1}, c_s] + Q[d_r, c_{s+1}] + Q[d_r, c_0]Q[d_0, c_s] - Q[d_r, d_0]Q[c_0, c_s]$$

Derivatives of a quasi-Pfaffian

Derivatives

$$
\begin{aligned}
Q[d_0, c_0]_x = Q[d_1, c_0] + Q[d_0, c_1]& \\
+ Q[d_0, c_0]Q[d_0, c_0] - {\color{red} Q[d_0, d_0]Q[c_0, c_0]}&
\end{aligned}
$$

$$
\begin{aligned}
Q[d_0, c_0]_{xx} = Q[d_2, c_0] + 2Q[d_1, c_1] + Q[d_0, c_2]& \\
+ Q[d_1, c_0]Q[d_0, c_0] + Q[d_0, c_0]Q[d_0, c_1]& \\
+ (Q[d_1, c_0] + Q[d_0, c_1] + Q[d_0, c_0]Q[d_0, c_0])Q[d_0, c_0]& \\
+ Q[d_0, c_0](Q[d_1, c_0] + Q[d_0, c_1] + Q[d_0, c_0]Q[d_0, c_0])&
\end{aligned}
$$

$$
\begin{aligned}
&Q[d_0, c_0]_{xxx} = \cdots \\
&Q[d_0, c_0]_{xxxx} = \cdots \\
&Q[d_0, c_0]_{y} = \cdots \\
&Q[d_0, c_0]_{yy} = \cdots \\
&\text{etc}
\end{aligned}
$$

What am I going to do with all these expressions?

$$w = 2Q[c_0, d_0], \qquad v = Q[c_0, c_1], \qquad \bar{v} = Q[d_0, d_1]$$

Calculate all necessary derivatives and sub into the equations.
In the commutative case we will find that the equations are satisfied. For the non-commutative case we will be able to see how to adapt the equations so that they are satisfied.

Non-commutative case

Set

$$w = 2Q[d_0, c_0], \qquad v = Q[c_0, c_1], \qquad \bar{v} = Q[d_0, d_1].$$

Also setting

$$Q[c_0, c_0] = Q[d_0, d_0] = 0$$

We find:

$$(4w_t - 3(w_x)^2 - w_{xxx})_x - 3w_{yy} + 24(v\bar{v})_x - 6(w_y w_x - w_x w_y) = 0$$

$$2v_t + 3w_x v_x + v_{xxx} + 3\left(v_{xy} + \frac{1}{2}(vw_y + w_y v)\right) - \frac{3}{2}(vw_{xx} - w_{xx}v) - \frac{3}{2}(w_x v - vw_x)w = 0$$

$$2\bar{v}_t + 3\bar{v}_x w_x + \bar{v}_{xxx} - 3\left(\bar{v}_{xy} + \frac{1}{2}(\bar{v}w_y + w_y\bar{v})\right) + \frac{3}{2}(\bar{v}w_{xx} - w_{xx}\bar{v}) - \frac{3}{2}w(w_x\bar{v} - \bar{v}w_x) = 0$$

Non-commutative case

If we **don't** set $Q[c_0, c_0] = Q[d_0, d_0] = 0$ but instead

$$J = Q[c_0, c_0], \quad H = Q[d_0, d_0],$$

we find our first equation is

$$\begin{aligned}
&(4w_t - 3(w_x)^2 - w_{xxx})_x - 3w_{yy} + 24(v\bar{v})_x - 6(w_y w_x - w_x w_y) - \\
&6(J_{xx}\bar{v} - J_y\bar{v} - J\bar{v}_{xx} - J\bar{v}_y - J_x H_{xx} - J_x H_y - J_{xx} H_x + J_y H_x) + \\
&6(vH_{xx} + vH_y - v_{xx}H + v_y H) + 12(vHw)_x \\
&+6((JHw)_y - JHw_{xx} - J_{xx}Hw) \\
&+6(JH_{xx}w - 2J_x H w_x) + 12(w_x J\bar{v} + wJ_x\bar{v} + wJ\bar{v}_x) + 6(wJ_{xx}H - \\
&wJH_{xx} - w_{xx}JH - 2w_x JH_x - (wJH)_y) + \\
&12(JHJH)_x - 12(wJHw)_x = 0
\end{aligned}$$

Non-commutative case

We find our second equation is

$$
\begin{aligned}
&2v_t + 3w_x v_x + v_{xxx} + 3\left(v_{xy} + \frac{1}{2}(vw_y + w_y v)\right) \\
&-\frac{3}{2}(vw_{xx} - w_{xx}v) - \frac{3}{2}(w_x v - vw_x)w - 3J_y\bar{w}_x + 3J_{xx}\bar{w}_x \\
&+2w_{xxx}J - 2w_t J + 3w_y J_x + 3w_{xx}J_x - \\
&6J\bar{w}_x\bar{w}_x - 3ww_y J + 3ww_{xx}J + 3w_y wJ + 3w_{xx}wJ + 6ww_x J_x + \\
&6w_x w_x J - 6J_x H_x J - 6J_x HJ_x + 6J_x Hv - 6v_x HJ - \\
&6J\bar{v}_x J - 6J\bar{v}J_x + 6J\bar{v}v - 6v\bar{v}J + 6w_x J\bar{w}_x + 6J_x\bar{w}_x \\
&-6J\bar{w}_x HJ - 6JHJ\bar{w}_x - 6J_x HwJ + 6JH_x wJ \\
&+6JHwJ_x + 6vHwJ - 6JHwv - 6J\bar{v}wJ + \\
&6(wJHv - wvHJ - wJHJ_x - wJH_x J + wJ_x HJ + \\
&ww J\bar{w}_x + ww_x wJ + wJ\bar{v}J) + \\
&6(JHwwJ - 2wJHwJ + wwJHJ) = 0
\end{aligned}
$$

Non-commutative case

Set

$$w = 2Q[d_0, c_0], \qquad v = Q[c_0, c_1], \qquad \bar{v} = Q[d_0, d_1].$$

Also setting

$$Q[c_0, c_0] = Q[d_0, d_0] = 0$$

We find:

$$(4w_t - 3(w_x)^2 - w_{xxx})_x - 3w_{yy} + 24(v\bar{v})_x - 6(w_y w_x - w_x w_y) = 0$$

$$2v_t + 3w_x v_x + v_{xxx} + 3\left(v_{xy} + \frac{1}{2}(vw_y + w_y v)\right)$$
$$- \frac{3}{2}(vw_{xx} - w_{xx}v) - \frac{3}{2}(w_x v - vw_x)w = 0$$

$$2\bar{v}_t + 3\bar{v}_x w_x + \bar{v}_{xxx} - 3\left(\bar{v}_{xy} + \frac{1}{2}(\bar{v}w_y + w_y\bar{v})\right)$$
$$+ \frac{3}{2}(\bar{v}w_{xx} - w_{xx}\bar{v}) - \frac{3}{2}w(w_x\bar{v} - \bar{v}w_x) = 0$$

Lax System (Adler & Postnikov)

$$\left(\partial_x^2 - \partial_y + w_x\right)\psi + 2\bar{v}\phi = 0$$
$$\left(\partial_x^2 + \partial_y + w_x\right)\phi + 2v\psi = 0$$
$$\left(\partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}(w_{xx} + w_y) - \partial_t\right)\psi + 3\bar{v}_x\phi = 0$$
$$\left(\partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}(w_{xx} + w_y) - \partial_t\right)\psi + 3\bar{v}_x\phi = 0$$

Consistency gives

$$(4w_t - 3(w_x)^2 - w_{xxx})_x - 3w_{yy} + 24(v\bar{v})_x - 6(w_y w_x - w_x w_y) = 0$$
$$2v_t + 3w_x v_x + v_{xxx} + 3\left(v_{xy} + \frac{1}{2}(vw_y + w_y v)\right) - \frac{3}{2}(vw_{xx} - w_{xx}v) = 0$$

Conclusions

I've introduced the idea of a quasi-Pfaffian and considered some ideas

- Like normal quasi-determinants There exist quasi-pfaffian identities
- Differentiation of quasi-pfaffians give very simple things
- We can cast the solutions of the Hirota Ohta system as quasi-pfaffians.
- In the fully non-commutative case we can see what adaptions are needed to the equations (add in extra commutators etc) to make them satisfied by quasi-pfaffians.