Noncommutative martingale inequalities: some recent progress II

Adam Osękowski

University of Warsaw, Poland

Connections between commutative
and non-commutative harmonic analysis
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Joint works with T. Gałązka, Y. Jiao, L. Wu and Y. Zuo

Plan

I. Inequalities for dominated martingales

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II. Weak L^∞ estimates for martingales

- $(\mathcal{M}, \tau)$ - von Neumann algebra with a normalized trace.

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· $(\mathcal{E}_n)_{n\geq 0}$ - the corresponding conditional expectations.

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- Define the associate differences

$$dx_0 = x_0, \quad dx_n = x_n - x_{n-1}, \quad n \geq 1 \quad \leftrightarrow \quad x_N = \sum_{n=0}^{N} dx_n.$$

General question

Let $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ be two $(\mathcal{M}_n)_{n\geq 0}$-martingales.

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Goal: Prove L^p and/or weak L^p estimates between x and y:

$$\|y_n\|_{L^p(\mathcal{M})} \lesssim_p \|x_n\|_{L^p(\mathcal{M})}, \qquad \|y_n\|_{L^{p,\infty}(\mathcal{M})} \lesssim_p \|x_n\|_{L^p(\mathcal{M})},$$

for $n = 0, 1, 2, \ldots$ and $1 \leq p \leq \infty$.

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What about the noncommutative context?

Example: martingale transforms

Suppose that $dy_n = v_n dx_n$, where for each n, v_n belongs to $\mathcal{M}_{n-1} \cap \mathcal{M}_n'$ and satisfies $\|v_n\|_{L^\infty} \leq 1$.

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Theorem (Burkholder 1966, classical case)

For any $1 < p < \infty$ there is a finite constant c_p such that

$$\|y_n\|_p \leq c_p \|x_n\|_p, \qquad n = 0, 1, 2, \ldots.$$

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Theorem (Pisier-Xu 1997, noncommutative case)

For any $1 < p < \infty$ there is a finite constant c_p such that

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Analogous weak-type bounds for $p = 1$.

Another example: differential subordination

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Suppose that $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ *are martingales such that* y *is differentially subordinate to* x. *Then for* $1 < p < \infty$,

$$||y_n||_p \leq \frac{Cp^2}{p-1}||x_n||_p, \qquad n = 0, 1, 2, \ldots$$

for some universal constant C. *Furthermore,*

$$||y_n||_{1,\infty} \leq c||x_n||_1, \qquad n = 0, 1, 2, \ldots.$$

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Theorem (Jiao, O., Wu '18)

Let $2 \leq p < \infty$. Suppose that $x = (x_n)_{n \geq 0}$, $y = (y_n)_{n \geq 0}$ are martingales such that y is differentially subordinate to x. Then

$$||y_n||_{L^p(\mathcal{M})} \leq Cp||x_n||_{L^p(\mathcal{M})}, \qquad n = 0, 1, 2, \ldots,$$

for some universal C. For $1 < p < 2$, the inequality fails to hold.

Another example: differential subordination

Definition (noncommutative: Jiao, O., Wu '18)

A martingale y is strongly differentially subordinate to x, if for any n and any projection $R \in \mathcal{M}_{n-1}$, we have

$$Rdy_nRdy_nR \leq Rdx_nRdx_nR.$$

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Theorem (Jiao, O., Wu '18)

Suppose that $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ are martingales such that y is strongly differentially subordinate to x. Then for $1 < p < 2$,

$$||y_n||_{L^p(\mathcal{M})} \leq \frac{C}{p-1}||x_n||_{L^p(\mathcal{M})}, \qquad n = 0, 1, 2, \dots,$$

for some universal C. Furthermore, we have

$$||y_n||_{L^{1,\infty}(\mathcal{M})} \leq c||x_n||_{L^1(\mathcal{M})}, \qquad n = 0, 1, 2, \dots.$$

Tangent martingales/weak domination

Definition (classical)

Martingales $x = (x_n)_{n\geq 0}$, $y = (y_n)_{n\geq 0}$ are said to be tangent, if for any n and any function $\varphi : \mathbb{R} \to \mathbb{R}$ we have

$$\mathcal{E}_{n-1}\varphi(dy_n) = \mathcal{E}_{n-1}\varphi(dx_n).$$

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Tangency $\Rightarrow$ weak domination.

Tangent martingales/weak domination

Theorem (classical: O. 2004)

Suppose that y is weakly dominated by x. Then

$$\|y_n\|_{1,\infty} \leq 3\|x_n\|_1, \qquad n = 0, 1, 2, \ldots,$$

and for $1 < p < \infty$,

$$\|y_n\|_p \leq \frac{Cp^2}{p-1}\|x_n\|_p, \qquad n = 0, 1, 2, \ldots.$$

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Actually, it suffices to assume $\mathcal{E}_{n-1}\varphi(|dy_n|) \leq \mathcal{E}_{n-1}\varphi(|dx_n|)$ for:

· $p \geq 2$: $\varphi(x) = x^2$ and $\varphi(x) = x^p$;

· $p < 2$: $\varphi(x) = \psi(\lambda x)$, $\lambda > 0$; for some $\psi(x) \sim \min\{x^2, x^p\}$.

Theorem (Jiao, O., Wu, 2019)

Suppose that for any n,

$$\mathcal{E}_{n-1}(dy_n^2) \leq \mathcal{E}_{n-1}(dx_n^2), \qquad \mathcal{E}_{n-1}(dy_n^p) \leq \mathcal{E}_{n-1}(dx_n^p).$$

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Noncommutative weak domination

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Theorem (Jiao, O., Wu, Zuo 2023)

Suppose that for any n, any projection $R \in \mathcal{M}_{n-1}$ and any $\lambda > 0$,

$$\mathcal{E}_{n-1}\varphi(\lambda R dy_n R) \leq \mathcal{E}_{n-1}\varphi(\lambda R dx_n R),$$

where $\varphi(x) = \min\{x^2, x^p\}$. *Then for* $1 < p < 2$,

$$\|y_n\|_{L^p(\mathcal{M})} \leq \frac{C}{p-1}\|x_n\|_{L^p(\mathcal{M})}, \qquad n = 0,\ 1,\ 2,\ \ldots.$$

Application: free Hilbert transform

Let $G = \mathbb{F}_q$ be the free group of q generators g_1, g_2, ..., g_q.

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Following Pisier, Biane, Ozawa, Mei–Ricard, . . . we let

$$f = \sum_{g \in G} \hat{f}(g)\lambda(g) \quad \to \quad \mathbb{H}f = \sum_{g \in G} \varepsilon(g)\hat{f}(g)\lambda(g),$$

where ε is the sign function given by

$$\varepsilon(g) = \begin{cases} \varepsilon_j^+ & \text{if } g \text{ starts with } g_j, \\ \varepsilon_j^- & \text{if } g \text{ starts with } g_j^{-1}, \end{cases}$$

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Theorem

For $p \geq 2$ we have

$$\|\mathbb{H}\|_{\mathcal{L}^p(\mathbb{F}_q) \to \mathcal{L}^p(\mathbb{F}_q)} \leq Cp \log p.$$

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If $u = \mathcal{P}[f]$, $v = \mathcal{P}\left[\mathcal{H}^{\mathbb{T}} f\right]$ are the Poisson extensions of f and $\mathcal{H}^{\mathbb{T}} f$ to the unit disc, then u, v enjoy C-R equations and $v(0) = 0$.

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Martingale L^p estimate $\Rightarrow$ L^p bound for $\mathcal{H}^{\mathbb{T}}$.

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$$X_t \sim \xi_t \rtimes f, \qquad Y_t \sim \zeta_t \rtimes \mathbb{H}f,$$

for some commutative martingales ξ, ζ.

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It remains to check that Y is dominated by X.

Application: Hilbert transform on a quantum torus

· Let $d \geq 2$ be a fixed dimension and let $\theta = (\theta_{kj})_{1 \leq j,k \leq d}$ be a real skew-symmetric $d \times d$ matrix.

· d-dimensional noncommutative torus $\mathcal{A}_\theta$: the universal C^*-algebra generated by the collection of d unitary operators U_1, U_2, ..., U_d which satisfy the commutation relation

$$U_k U_j = e^{2\pi i \theta_{kj}} U_j U_k, \qquad j, k = 1, 2, \ldots, d.$$

· for any polynomial

$$x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m, \qquad \alpha_m \in \mathbb{C},$$

we define $\tau(x) = \alpha_0$: it extends to a faithful tracial state on $\mathcal{A}_\theta$.

· quantum torus $\mathbb{T}^d_\theta$: the w^*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ.

Application: Hilbert transform on a quantum torus

For a given nonzero vector $a \in \mathbb{R}^d$, we define the Hilbert transform in the direction a by the formula

$$\mathcal{H}^a\left(\sum_m \alpha_m U^m\right) = \sum_m \mathrm{sgn}\langle m, a\rangle \alpha_m U^m,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$.

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Theorem (Chen, Xu, Yin 2013)

For $1 < p < \infty$ we have

$$\|\mathcal{H}^a\|_{L^p(\mathbb{T}^d_\theta) \to L^p(\mathbb{T}^d_\theta)} \leq \frac{Cp^2}{p-1}.$$

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Theorem (Jiao, O., Wu, Zuo 2023)

We have

$$\|\mathcal{H}^a\|_{L^1(\mathbb{T}^d_\theta) \to L^{1,\infty}(\mathbb{T}^d_\theta)} < \infty.$$

II. Weak L^∞ estimates for martingales

Theorem

Let T be a sublinear operator defined on simple functions f on a measure space (X, μ), taking values in measurable functions on (Y, ν). Let $1 \leq p_0 < p_1 < \infty$. Suppose that

- *T is of weak type (p_0, p_0) with constant A_0;*
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Then for every $p \in (p_0, p_1)$, T is of strong type (p, p), i.e.,

$$\|Tf\|_{L^p(Y)} \leq C_p \|f\|_{L^p(X)},$$

where C_p depends only on p_0, p_1, p, A_0, A_1, and not on f.

Theorem

Let T be a sublinear operator defined on simple functions f on a measure space (X, μ), taking values in measurable functions on (Y, ν). Let $1 \leq p_0 < p_1 < \infty$. Suppose that

- *T is of weak type (p_0, p_0) with constant A_0;*
- *T is of weak type (p_1, p_1) with constant A_1;*

Then for every $p \in (p_0, p_1)$, T is of strong type (p, p), i.e.,

$$\|Tf\|_{L^p(Y)} \leq C_p \|f\|_{L^p(X)},$$

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This works for $p_1 = \infty$ as well, but then $L^{p_1,\infty} = L^\infty$.

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Is there any variant of a weak-L^∞ space?

Weak L^∞

For a measurable operator x, we define its distributional function by

$$\lambda(s,x)=\tau(\chi_{(s,\infty)}(|x|)), \qquad s>0,$$

and the associated generalized singular numbers are

$$\mu(t,x)=\inf\Big\{s>0:\lambda(s,x)\leq t\Big\}, \qquad t>0.$$

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Motivated by Bennett, DeVore and Sharpley (1981), we let

$$wL^\infty(\mathcal{M})=\left\{x\,:\,\|x\|_{wL^\infty(\mathcal{M})}=\sup_{t>0}\Big(\frac{1}{t}\int_0^t\mu(s,x)ds-\mu(t,x)\Big)<\infty\right\}.$$

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We have $wL^\infty(\mathcal{M})\supset\mathcal{M}$; $wL^\infty(\mathcal{M})$ is not a linear space.

Theorem (Bennett, DeVore, Sharpley 1981; Jiao, O., Wu, Zuo 2025)

Fix $1 \leq p_0 < \infty$. Suppose that T is a sublinear operator such that for any projection e with $\tau(e) < \infty$ we have

$$\|Te\|_{L^{p_0,\infty}(\mathcal{M})} \leq \|e\|_{L^{p_0}(\mathcal{M})} \qquad \text{and} \qquad \|Te\|_{wL^\infty(\mathcal{M})} \leq \|e\|_{L^\infty(\mathcal{M})}.$$

Then for any $p_0 < p < \infty$ and any simple operator f,

$$\|Tf\|_{L^p(\mathcal{M})} \leq \frac{Cp^2}{p - p_0}\|f\|_{L^p(\mathcal{M})},$$

for some universal constant C. Furthermore, if T is linear, then T can be uniquely extended to a bounded linear operator on $L^p(\mathcal{M})$.

We define the martingale column *BMO* space

$$BMO^c(\mathcal{M})$$
$$= \Big\{ x \in L^2(\mathcal{M}) \,:\, \|x\|_{BMO^c(\mathcal{M})} = \sup_{n\geq 0} \big\| \mathcal{E}_n(|x - \mathcal{E}_{n-1}(x)|^2) \big\|_\infty^{1/2} < \infty \Big\}.$$

The associated row space

$$BMO^r(\mathcal{M}) = \Big\{ x \in L^2(\mathcal{M}) \,:\, \|x\|_{BMO^r(\mathcal{M})} = \|x^*\|_{BMO^c(\mathcal{M})} < \infty \Big\}.$$

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The two-sided BMO class: $BMO(\mathcal{M}) = BMO^c(\mathcal{M}) \cap BMO^r(\mathcal{M})$, equipped with the norm

$$\|x\|_{BMO(\mathcal{M})} = \max\left\{\|x\|_{BMO^c(\mathcal{M})}, \|x\|_{BMO^r(\mathcal{M})}\right\}.$$

Theorem (Jiao, O., Wu, Zuo 2025)

For any $x \in BMO(\mathcal{M})$, we have

$$\|x\|_{wL^\infty(\mathcal{M})} \leq 16\|x\|_{BMO(\mathcal{M})}$$

and

$$\|\mu(\cdot, x)\|_{BMO(\mathbb{R}_+)} \leq 64\|x\|_{BMO(\mathcal{M})}.$$

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Key distributional estimate.

Theorem

Let $x = (x_n)_{n=0}^N$ be a finite self-adjoint martingale with $\|x_N\|_{BMO(\mathcal{M})} \leq 1$. Then for $\alpha > 0$ we have

$$\tau\big(|x_N|\chi_{(\alpha,\infty)}(|x_N|)\big) \leq (\alpha + 16)\tau\big(\chi_{(\alpha,\infty)}(|x_N|)\big).$$

Idea of proof

Key distributional estimate.

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Proof: Cuculescu projections and manipulations.

Application: Stein inequality

Theorem (Pisier-Xu 1997, Randrianantoanina 2002)

For $1 < p < \infty$ *and a finite sequence* $(a_n)_{n\geq 0}$ *in* $L^p(\mathcal{M})$,

$$\Big\| \Big(\sum_{n\geq 0} |\mathcal{E}_n(a_n)|^2 \Big)^{1/2} \Big\|_{L^p(\mathcal{M})} \leq C_p \Big\| \Big(\sum_{n\geq 0} |a_n|^2 \Big)^{1/2} \Big\|_{L^p(\mathcal{M})}.$$

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Theorem (Jiao, O., Wu, Zuo)

For a finite sequence $(a_n)_{n\geq 0}$ *of operators in* $L^1(\mathcal{M})$,

$$\Big\|\Big(\sum_{n\geq 0}|\mathcal{E}_n(a_n)|^2\Big)^{1/2}\Big\|_{wL^\infty(\mathcal{M})} \leq 32\Big\|\Big(\sum_{n\geq 0}|a_n|^2\Big)^{1/2}\Big\|_{L^\infty(\mathcal{M})},$$

$$\Big\|\mu\Big(\cdot,\Big(\sum_{n\geq 0}|\mathcal{E}_n(a_n)|^2\Big)^{1/2}\Big)\Big\|_{BMO(\mathbb{R}_+)} \leq 128\Big\|\Big(\sum_{n\geq 0}|a_n|^2\Big)^{1/2}\Big\|_{L^\infty(\mathcal{M})}.$$

Application: wL^∞ estimate for operator-valued functions

For an operator-valued function $f : \mathbb{R}^d \to \mathcal{M}$, let

$$\|f\|_{BMO^c(\mathbb{R}^d,\mathcal{M})} = \sup_{Q\subset\mathbb{R}^d} \left\| \left(\frac{1}{|Q|} \int_Q |f - f_Q|^2 \right)^{1/2} \right\|_{L^\infty(\mathcal{M})}.$$

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Theorem (Jiao, O., Wu, Zuo)

For any operator-valued function $f\in BMO(\mathbb{R}^d,\mathcal{M})$ satisfying $\|\frac{1}{|Q|}\int_Q f\|_{L^\infty(\mathcal{M})}\to 0$ as $|Q|\to\infty$, we have

$$\|f\|_{wL^\infty(\mathbb{R}^d,\mathcal{M})}\leq C_d\|f\|_{BMO(\mathbb{R}^d,\mathcal{M})}.$$

and

$$\|\mu(\cdot,f)\|_{BMO(\mathbb{R}_+)}\leq C_d\|f\|_{BMO(\mathbb{R}^d,\mathcal{M})},$$

for some universal constant C_d depending on the dimension only.

Thank you for your attention!