Vectorial Darboux Transformations

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To the memory of

Jonathan Nimmo
25th July 1958 – 20th June 2017

Aristophanes Dimakis
24th March 1953 – 8th July 2021

Vectorial Binary Darboux Transformation

Nonlinear equation for ϕ (from a certain class)

seed solution ϕ
($m \times n$)

$\swarrow$ $\searrow$

linear system
for θ ($m \times N$)

adjoint linear system
for η ($N \times n$)

$\searrow$ $\swarrow$

compatible linear eqs
including *Sylvester eq.*
for "Darboux potential" Ω

$\downarrow$

new solution

$$\boxed{\phi' = \phi - \theta\, \Omega^{-1}\, \eta}$$

A. Sakhnovich 1994, M. Mañas 1996, A. Dimakis and M-H 2013

Precessor: Marchenko, *Nonlinear Equations and Operator Algebras*, '87

Comparison

Classical bDT

- The linear system and the "adjoint" are Lax pairs.
- The (adj.) linear system depends on "spectral parameter" γ (δ) and on seed solution.
- Solution of lin. systems, for fixed parameters, is used to determine a single soliton solution of the ϕ-equation.
- This bDT can be iterated with N *different* spectral parameters to generate an N-soliton solution.
- Via special limits, other classes of solutions, like "multiple pole solutions" (e.g. higher order rogue waves) can be reached.

Vectorial bDT

- Linear systems are N-vectorial.
- Spectral parameters become $N \times N$ "spectral matrices" Γ, Δ.
- No iteration or limits necessary. Ordinary N-solitons obtained from diagonal Γ and Δ with pairwise different eigenvalues.
- Multiple pole solutions: Γ and Δ Jordan blocks.
- Solutions corresp. to same seed solution can be superposed.

Vectorial binary Darboux transformation for (nc)AKNS(-1)

The latter is here our guiding example.

Example: Vectorial bDT for AKNS(-1)

Let a, f, u be a solution of the scalar AKNS(-1) (or CID) system

$$f_{xt} + 2a\, f = 0 \quad u_{xt} + 2a\, u = 0 \quad a_t = (f\, u)_x \quad \text{(AKNS(-1) or CID)}$$

Let θ_i (η_i), $i = 1, 2$, be N-component row (column) vector sol's of

$$\begin{aligned} \theta_{1x}\Delta &= -a\,\theta_1 - f_x\,\theta_2 & \theta_{2x}\Delta &= a\,\theta_2 - u_x\,\theta_1 \\ \theta_{1t} &= \frac{1}{2}\theta_1\Delta + f\,\theta_2 & \theta_{2t} &= -\frac{1}{2}\theta_2\Delta - u\,\theta_1 \\ \Gamma\,\eta_{1x} &= a\,\eta_1 + u_x\,\eta_2 & \Gamma\,\eta_{2x} &= -a\,\eta_2 + f_x\,\eta_1 \\ \eta_{1t} &= -\frac{1}{2}\Gamma\,\eta_1 + u\,\eta_2 & \eta_{2t} &= \frac{1}{2}\Gamma\,\eta_2 - f\,\eta_1 \end{aligned}$$

with constant invertible $N \times N$ matrices Δ, Γ. Let Ω solve

$$\boxed{\Gamma\,\Omega - \Omega\,\Delta = \sum_i \eta_i\,\theta_i} \quad \Omega_x\Delta = -\sum_i \eta_{ix}\theta_i \quad \Omega_t = -\frac{1}{2}\eta_1\theta_1 + \frac{1}{2}\eta_2\theta_2$$

$$\Longrightarrow \quad a' = a - (\theta_1\Omega^{-1}\eta_1)_x \qquad f' = f - \theta_1\Omega^{-1}\eta_2 \qquad u' = u - \theta_2\Omega^{-1}\eta_1$$

is also a solution of the AKNS(-1) system.

Different types of solutions

If a linear system has constant coefficients, we have to distinguish the following cases:

I. General solution is linear combination of exponentials

II. Degenerate cases $\curvearrowright$ possibility of quasi-rational solutions of nonlinear equation (e.g. *rogue waves*)

1. Spectrum condition holds: $\mathrm{spec}(\Gamma) \cap \mathrm{spec}(\Delta) = \emptyset$.

The Sylvester equation then has a unique solution Ω.

- Diagonal Γ, Δ → "ordinary solitons" ($\curvearrowleft$ classical DT)
- Γ, Δ Jordan blocks → "multiple pole solutions"

2. Spectrum condition violated.

- There are constraints on solutions of linear systems
- various cases, differential eqs for Ω have to be integrated $\curvearrowright$ new dependence of sol. of nonl. eq. on independent variables

M-H & Ye 2025: extensive elaboration for vector Fokas-Lenells equation.

Example: A reduction of AKNS(-1)

Setting $u = \epsilon f^*$, $\epsilon \in \{\pm 1\}$, AKNS(-1) reduces to

$$f_{xt} + 2a\, f = 0 \qquad a_t = \epsilon\, (|f|^2)_x \qquad \text{(“CCD equations”)}$$

Implementation in bDT: $\theta_1 = \eta_1^\dagger \quad \theta_2 = \epsilon\, \eta_2^\dagger \quad \Delta = -\Gamma^\dagger \quad \Omega^\dagger = \Omega$

Sylvester equation now becomes Lyapunov equation

$$\Gamma\, \Omega + \Omega\, \Gamma^\dagger = \eta_1 \eta_1^\dagger + \epsilon\, \eta_2 \eta_2^\dagger$$

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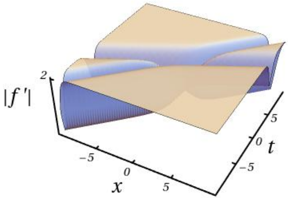
$$\Gamma\, \Omega + \Omega\, \Gamma^\dagger = \eta_1 \eta_1^\dagger + \epsilon\, \eta_2 \eta_2^\dagger$$

Example. Plane wave seed $\quad f = C\, e^{\mathrm{i}\,(\alpha\, x - \beta\, t)} \qquad a = -\frac{1}{2}\alpha\beta$
with $C \in \mathbb{C} \setminus \{0\}$, $\alpha, \beta \in \mathbb{R}$. If $\epsilon = -1$, solutions with diagonal Γ and spectrum condition are singular. But if all eigenvalues are imaginary, f' describes non-singular dark solitons.



(Feng, Ling, Zhu 2016: via limit procedure)

Superposition property

Let $(\Gamma^{(i)}, \Delta^{(i)}, \theta^{(i)}, \eta^{(i)}, \Omega^{(i)})$, $i = 1, 2$, be data that determine solutions of the respective nonlinear equation, obtained via a vectorial bDT from the same seed solution. Set $\theta = \begin{pmatrix} \theta^{(1)} & \theta^{(2)} \end{pmatrix}$ and

$$\Gamma = \begin{pmatrix} \Gamma^{(1)} & 0 \\ 0 & \Gamma^{(2)} \end{pmatrix}, \; \Delta = \begin{pmatrix} \Delta^{(1)} & 0 \\ 0 & \Delta^{(2)} \end{pmatrix}, \; \eta = \begin{pmatrix} \eta^{(1)} \\ \eta^{(2)} \end{pmatrix}, \; \Omega = \begin{pmatrix} \Omega^{(1)} & \Omega^{(12)} \\ \Omega^{(21)} & \Omega^{(2)} \end{pmatrix}$$

Then the linear systems are solved and it only remains to determine the matrices $\Omega^{(12)}$ and $\Omega^{(21)}$ in order to obtain a new solution of the nonlinear equation. The Sylvester equation reduces to the two Sylvester equations

$$\Gamma^{(1)}\Omega^{(12)} - \Omega^{(12)}\Delta^{(2)} = \eta^{(1)}\,\theta^{(2)}$$
$$\Gamma^{(2)}\Omega^{(21)} + \Omega^{(21)}\Delta^{(1)} = \eta^{(2)}\,\theta^{(1)}$$

If the corresponding spectrum conditions are satisfied, these equations have unique solutions $\Omega^{(12)}$ and $\Omega^{(21)}$. (Otherwise additional equations have to be solved.)

Example: Vectorial bDT for ncAKNS(-1)

Let $\mathcal{A}$ be an associative algebra and $f, u, a, \tilde{a} \in \mathcal{A}$ a solution of

$$f_{xt} = -a\,f - f\,\tilde{a} \qquad u_{xt} = -\tilde{a}\,u - u\,a \qquad a_t = (fu)_x \qquad \tilde{a}_t = (uf)_x$$

Let N-component row, resp. column vectors (over $\mathcal{A}$), θ_i, η_i, solve

$$\begin{aligned}
\theta_{1x}\Delta &= -a\,\theta_1 - f_x\,\theta_2 & \theta_{2x}\Delta &= -u_x\,\theta_1 + \tilde{a}\,\theta_2 \\
\theta_{1t} &= \frac{1}{2}\theta_1\,\Delta + f\,\theta_2 & \theta_{2t} &= -u\,\theta_1 - \frac{1}{2}\theta_2\,\Delta \\
\Gamma\,\eta_{1x} &= \eta_1\,a + \eta_2\,u_x & \Gamma\,\eta_{2x} &= \eta_1\,f_x - \eta_2\,\tilde{a} \\
\eta_{1t} &= -\frac{1}{2}\Gamma\,\eta_1 + \eta_2\,u & \eta_{2t} &= -\eta_1\,f + \frac{1}{2}\Gamma\,\eta_2
\end{aligned}$$

with constant $N \times N$ matrices Δ, Γ. Let Ω solve

$$\boxed{\Gamma\Omega - \Omega\Delta = \sum_i \eta_i\,\theta_i} \qquad \Omega_x\Delta = -\sum_i \eta_{ix}\,\theta_i \quad \Omega_t = -\frac{1}{2}(\eta_1\theta_1 - \eta_2\theta_2)$$

$$\Rightarrow \qquad f' = f - \theta_1\Omega^{-1}\eta_2 \qquad u' = u - \theta_2\Omega^{-1}\eta_1 \qquad a' = a - (\theta_1\Omega^{-1}\eta_1)_x$$

$$\tilde{a}' = \tilde{a} + (\theta_2\Omega^{-1}\eta_2)_x$$

is a new solution.

Remark

Choosing for $\mathcal{A}$ the algebra of 2×2 matrices of the form

$$f = \begin{pmatrix} f_1 & f_2 \\ -f_2^* & f_1^* \end{pmatrix} \qquad \text{representation of } \mathbb{H}$$

with complex functions f_1, f_2, and imposing the reduction condition $u = \epsilon f$, $\tilde{a} = a$, we obtain a 2-component vector version of the CCD system:

$$f_{\mu,xt} + 2a\, f_\mu = 0 \qquad \mu = 1, 2 \qquad a_t = \epsilon\, (\|f\|^2)_x$$

where a is now taken to be a real scalar (since the matrix a turns out to be a real scalar times the identity matrix).

Remark

The ncAKNS(-1) system arises from

$$\phi_{xt} = \frac{1}{2}\left[\,[J,\phi]\,,\,\phi_x - \frac{1}{2}J\right]$$

with

$$\phi = \begin{pmatrix} p & f \\ u & -\tilde{p} \end{pmatrix}$$

and setting $J = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$, or equivalently $J = 2\mathcal{P}$ with $\mathcal{P} = \mathrm{diag}(\mathbf{1}, 0)$, and $a = p_x - \frac{1}{2}\mathbf{1}$, $\tilde{a} = \tilde{p}_x - \frac{1}{2}\mathbf{1}$.

The above equation results from

$$\boxed{\bar{\mathrm{d}}\,\mathrm{d}\,\phi = \mathrm{d}\phi \wedge \mathrm{d}\phi}$$

by setting, for any 2×2 matrix F over $\mathcal{A}$,

$$\mathrm{d}F = \frac{1}{2}[J, F]\,\xi + [\partial_x, F]\,\bar{\xi} \qquad \bar{\mathrm{d}}F = [\partial_t, F]\,\xi + \frac{1}{2}[J, F]\,\bar{\xi}$$

where ξ, $\bar{\xi}$ are constant Grassmann numbers. (Dimakis & M-H '10)

Binary Darboux transformation in bidifferential calculus

Bidifferential calculus

Bidifferential calculus $(\Omega(\mathcal{A}), \mathrm{d}, \bar{\mathrm{d}})$ over an associative algebra $\mathcal{A}$

- graded algebra $\Omega(\mathcal{A}) = \bigoplus_{n \geq 0} \Omega^k(\mathcal{A})$, $\qquad \Omega^0(\mathcal{A}) = \mathcal{A}$
- two linear maps $\mathrm{d}, \bar{\mathrm{d}} : \Omega^k(\mathcal{A}) \longrightarrow \Omega^{k+1}(\mathcal{A})$ such that

$$\mathrm{d}^2 = 0 \qquad \bar{\mathrm{d}}^2 = 0 \qquad \mathrm{d}\bar{\mathrm{d}} + \bar{\mathrm{d}}\mathrm{d} = 0$$

  and $\mathrm{d}, \bar{\mathrm{d}}$ satisfy the graded *Leibniz rule*

For a 1-form A, $\boxed{\mathrm{d}A = 0 \quad \bar{\mathrm{d}}A = A^2}$ is then *integrable*. If

$$A = \mathrm{d}\phi \quad \text{respectively} \quad A = (\bar{\mathrm{d}}g)\, g^{-1}$$

then we have

$$\bar{\mathrm{d}}\, \mathrm{d}\, \phi = \mathrm{d}\phi\, \mathrm{d}\phi \qquad \text{respectively} \qquad \mathrm{d}\, [(\bar{\mathrm{d}}g)\, g^{-1}] = 0$$

Features:

- generalization of integrable structure of *sdYM*
- Miura transformation: $\mathrm{d}\phi = (\bar{\mathrm{d}}g)\, g^{-1}$
- also covers nc*KP* (hierarchy) and *discrete* nc integrable eqs

Binary Darboux Transformation in Bidifferential Calculus

Given a bidiff. calculus, let 0-forms Δ, Γ and 1-forms κ, λ satisfy

$$\bar{\mathrm{d}}\Delta + [\lambda, \Delta] = (\mathrm{d}\Delta)\,\Delta \qquad \bar{\mathrm{d}}\lambda + \lambda^2 = (\mathrm{d}\lambda)\,\Delta$$
$$\bar{\mathrm{d}}\Gamma - [\kappa, \Gamma] = \Gamma\,\mathrm{d}\Gamma \qquad \bar{\mathrm{d}}\kappa - \kappa^2 = \Gamma\,\mathrm{d}\kappa$$

Let 0-forms θ and η be solutions of the linear equations

$$\bar{\mathrm{d}}\theta = A\,\theta + (\mathrm{d}\theta)\,\Delta + \theta\,\lambda \qquad \bar{\mathrm{d}}\eta = -\eta\,A + \Gamma\,\mathrm{d}\eta + \kappa\,\eta$$

where the 1-form A satisfies

$$\mathrm{d}A = 0 \qquad \bar{\mathrm{d}}A = A^2$$

Furthermore, let Ω be an invertible solution of the linear system

$$\Gamma\,\Omega - \Omega\,\Delta = \eta\,\theta$$
$$\bar{\mathrm{d}}\Omega = (\mathrm{d}\Omega)\,\Delta - (\mathrm{d}\Gamma)\,\Omega + \kappa\,\Omega + \Omega\,\lambda + (\mathrm{d}\eta)\,\theta$$

Then $A' := A - \mathrm{d}(\theta\,\Omega^{-1}\eta)$ also solves the above equations.

Example: Self-dual Yang-Mills

$$\mathrm{d}f = f_y\,\xi_1 + f_z\,\xi_2 \qquad \bar{\mathrm{d}}f = f_{\bar{z}}\,\xi_1 + f_{\bar{y}}\,\xi_2 \qquad \mathcal{A} = C^\infty(\mathbb{R}^4)$$

We recover two well-known potential forms of the sdYM equation:

$$\bar{\mathrm{d}}\,\mathrm{d}\,\phi = \mathrm{d}\phi \wedge \mathrm{d}\phi \quad \Longleftrightarrow \quad \phi_{\bar{y}y} - \phi_{\bar{z}z} + [\phi_y, \phi_z] = 0$$

$$\mathrm{d}\,[(\bar{\mathrm{d}}g)\,g^{-1}] = 0 \quad \Longleftrightarrow \quad (g_{\bar{y}}\,g^{-1})_y - (g_{\bar{z}}\,g^{-1})_z = 0$$

Solutions in Nimmo, Gilson & Ohta (2000) are recovered by bDT.
In addition:
solutions with *non-constant* Γ, Δ, subject to matrix Hopf equations (Zenchuk 2008, Chvartatskyi, M-H, Stoilov 2015).

The latter type of solutions is important in the reduction of sdYM to the non-autonomous chiral model arising from the axially symmetric stationary vacuum Einstein (-Maxwell) equations ($\curvearrowright$ rotating black holes on an axis) and higher-dimensional counter parts. See Dimakis & M-H 2013.

Extension of ncAKNS(-1) to a hierarchy

$\mathcal{A}$ unital associative algebra, elements depend on independent variables $\boldsymbol{t} = (t_1, t_2, \ldots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$. Miwa shift operators:

$$\mathbb{E}_\nu F =: F_{[\nu]}\, \mathbb{E}_\nu \qquad \mathbb{E}_\nu^{-1} F =: F_{-[\nu]}\, \mathbb{E}_\nu^{-1}$$

$$F_{\pm[\nu]}(t_1, t_2, t_3, \ldots, \bar{\boldsymbol{t}}) := F(t_1 \pm \nu, t_2 \pm \nu^2/2, t_3 \pm \nu^3/3, \ldots, \bar{\boldsymbol{t}})$$

$$F_{\pm[\bar{\mu}]}(\boldsymbol{t}, \bar{t}_1, \bar{t}_2, \ldots) := F(\boldsymbol{t}, \bar{t}_1 \pm \mu, \bar{t}_2 \pm \mu^2/2, \ldots)$$

Let J be a constant $m \times m$ matrix and

$$\mathrm{d}F = \frac{1}{2}[J\,\mathbb{E}_\nu, F]\,\xi + \mu^{-1}[\mathbb{E}_{\bar{\mu}}, F]\,\bar{\xi} \qquad \bar{\mathrm{d}}F = \nu^{-1}[\mathbb{E}_\nu, F]\,\xi + \frac{1}{2}[J\,\mathbb{E}_{\bar{\mu}}, F]\,\bar{\xi}$$

Then $\quad \bar{\mathrm{d}}\,\mathrm{d}\,\phi = \mathrm{d}\phi \wedge \mathrm{d}\phi \quad$ (with ϕ $m \times m$) takes the form

$$\Big(\frac{1}{2}J - \mu^{-1}(\phi - \phi_{-[\bar{\mu}]})\Big)_{-[\nu]}\Big(\nu^{-1}\mathbf{1}_m - \frac{1}{2}(J\phi - \phi_{-[\nu]}J)\Big) = \Big(\nu^{-1}\mathbf{1}_m - \frac{1}{2}(J\phi - \phi_{-[\nu]}J)\Big)_{-[\bar{\mu}]}\Big(\frac{1}{2}J - \mu^{-1}(\phi - \phi_{-[\bar{\mu}]})\Big)$$

At $\mu^0\nu^0$ we recover the (generalized) ncAKNS(-1) equation.

$\mu^0 \nu^0$:

$$\phi_{t_1 \bar{t}_1} = \frac{1}{2}\Big[[J,\phi], \phi_{\bar{t}_1} - \frac{1}{2}J\Big]$$

$\mu^0 \nu^1$ (after use of the above):

$$\phi_{t_2 \bar{t}_1} = \frac{1}{4}[J^2, \phi_{t_1}] + \frac{1}{8}\{\phi, J[J,\phi]J\} - \frac{1}{8}J[J,\phi^2]J - \frac{1}{2}[\phi_{\bar{t}_1}, \{J, \phi_{t_1}\} + \frac{1}{2}[J,\phi]^2]$$

Compatibility condition $(\phi_{t_1 \bar{t}_1})_{t_2} = (\phi_{t_2 \bar{t}_1})_{t_1}$ for these flows:

$$\Big[\phi_{\bar{t}_1} - \frac{1}{2}J\,,\ [J,\phi_{t_2}] - \{J, \phi_{t_1 t_1}\} - J\phi_{t_1}\,[J,\phi] + [J,\phi]\,\phi_{t_1} J\Big] = 0$$

This is satisfied as a consequence of

$$[J,\phi_{t_2}] - \{J, \phi_{t_1 t_1}\} - J\phi_{t_1}\,[J,\phi] + [J,\phi]\,\phi_{t_1} J = 0$$

which does not belong to the hierarchy under consideration, but to the (“positive”) ncAKNS hierarchy ($\curvearrowright$ NLS system).

The big ncAKNS hierarchy

We complete the hierarchy by setting

$$\begin{aligned} \mathrm{d}F &= \frac{1}{2}[J\mathbb{E}_{\nu_1}, F]\,\xi_1 + \frac{1}{2}[J\mathbb{E}_{\nu_2}, F]\,\xi_2 + \mu_1^{-1}[\mathbb{E}_{\bar{\mu}_1}, F]\,\bar{\xi}_1 + \mu_2^{-1}[\mathbb{E}_{\bar{\mu}_2}, F]\,\bar{\xi}_2 \\ \bar{\mathrm{d}}F &= \nu_1^{-1}[\mathbb{E}_{\nu_1}, F]\,\xi_1 + \nu_2^{-1}[\mathbb{E}_{\nu_2}, F]\,\xi_2 + \frac{1}{2}[J\mathbb{E}_{\bar{\mu}_1}, F]\,\bar{\xi}_1 + \frac{1}{2}[J\mathbb{E}_{\bar{\mu}_2}, F]\,\bar{\xi}_2 \end{aligned}$$

where ξ_1, ξ_2, $\bar{\xi}_1$, $\bar{\xi}_2$ are linearly independent Grassmann numbers. Then $\bar{\mathrm{d}}\,\mathrm{d}\,\phi = \mathrm{d}\phi \wedge \mathrm{d}\phi$ additionally leads to

$$\begin{aligned} &\Big(\nu^{-1}\mathbf{1}_m - \frac{1}{2}(J\phi - \phi_{-[\nu]}J)\Big)_{-[\mu]}\Big(\mu^{-1}\mathbf{1}_m - \frac{1}{2}(J\phi - \phi_{-[\mu]}J)\Big) \\ &\qquad = \Big(\mu^{-1}\mathbf{1}_m - \frac{1}{2}(J\phi - \phi_{-[\mu]}J)\Big)_{-[\nu]}\Big(\nu^{-1}\mathbf{1}_m - \frac{1}{2}(J\phi - \phi_{-[\nu]}J)\Big) \\ &\Big(\frac{1}{2}J - \nu^{-1}(\phi - \phi_{-[\bar{\nu}]})_{-[\bar{\mu}]}\Big)\Big(\frac{1}{2}J - \mu^{-1}(\phi - \phi_{-[\bar{\mu}]})\Big) \\ &\qquad = \Big(\frac{1}{2}J - \mu^{-1}(\phi - \phi_{-[\bar{\mu}]})_{-[\bar{\nu}]}\Big)\Big(\frac{1}{2}J - \nu^{-1}(\phi - \phi_{-[\bar{\nu}]})\Big) \end{aligned}$$

and there is a corresponding bDT obtained from the general result of bidifferential calculus (which confirms that we have a hierarchy).

Vectorial bDT of the big ncAKNS hierarchy

$\mathcal{A}$ associative algebra, Δ, Γ constant invertible $N \times N$ matrices, ϕ a solution of big ncAKNS hierarchy. Let θ ($m \times N$), η ($N \times m$) solve

$$\frac{1}{\nu}(\theta - \theta_{-[\nu]}) = \frac{1}{2} J\theta\Delta + \frac{1}{2}(J\phi - \phi_{-[\nu]}J)\,\theta$$

$$\frac{1}{\mu}(\theta - \theta_{-[\bar{\mu}]})\,\Delta = \left(\frac{1}{2}J - \frac{1}{\mu}(\phi - \phi_{-[\bar{\mu}]})\right)\theta$$

and

$$\frac{1}{\nu}(\eta_{[\nu]} - \eta) = -\frac{1}{2}\eta\,(J\phi_{[\nu]} - \phi J) - \frac{1}{2}\Gamma\eta J$$

$$\Gamma\,\frac{1}{\mu}(\eta_{[\bar{\mu}]} - \eta) = \eta\left(\frac{1}{\mu}(\phi_{[\bar{\mu}]} - \phi) - \frac{1}{2}J\right)$$

Let Ω ($N \times N$) solve $\Gamma\,\Omega - \Omega\,\Delta = \eta\,\theta$ and

$$\frac{1}{\nu}(\Omega_{[\nu]} - \Omega) + \frac{1}{2}\eta J\theta_{[\nu]} = 0 \qquad \frac{1}{\mu}(\Omega_{[\bar{\mu}]} - \Omega)\,\Delta + \frac{1}{\mu}(\eta_{[\bar{\mu}]} - \eta)\,\theta_{[\bar{\mu}]} = 0$$

Then, wherever Ω is invertible,

$\Rightarrow \quad \phi' = \phi - \theta\,\Omega^{-1}\eta$ also solves the big ncAKNS hierarchy.

Remark

Choosing $J = 2\mathcal{P}$ and decomposing ϕ as before, we obtain in particular

$$
\begin{aligned}
&\left(\nu^{-1} I - p + p_{-[\nu]}\right)_{-[\mu]} \left(\mu^{-1} I - p + p_{-[\mu]}\right) - (fu)_{-[\mu]} \\
&\qquad = \left(\mu^{-1} I - p + p_{-[\mu]}\right)_{-[\nu]} \left(\nu^{-1} I - p + p_{-[\nu]}\right) - (fu)_{-[\nu]}
\end{aligned}
$$

Using $p_x = f\,u$ (which results from $\mu^0 \nu^1$), this becomes

$$
\begin{aligned}
&(\nu^{-1} I - p + p_{-[\nu]})_{-[\mu]}(\mu^{-1} I - p + p_{-[\mu]}) - (p_x)_{-[\mu]} \\
&\qquad = (\mu^{-1} I - p + p_{-[\mu]})_{-[\nu]}(\nu^{-1} I - p + p_{-[\nu]}) - (p_x)_{-[\nu]}
\end{aligned}
$$

which is a generating equation for the potential ncKP hierarchy (Bogdanov, Konopelchenko 1998).
There is a bidifferential calculus such that this is obtained from $\bar{\mathrm{d}}\mathrm{d}\phi = \mathrm{d}\phi \wedge \mathrm{d}\phi$, and there is a corresponding vectorial bDT.

Frölicher-Nijenhuis integrable systems

A special case of bidifferential calculus

Frölicher-Nijenhuis integrable systems

A class of bidifferential calculi is provided by Frölicher-Nijenhuis theory in differential geometry. Let d be the exterior derivative on a manifold $\mathcal{M}$. Associated with any tensor field N of type $(1,1)$,

$$N = N^{\mu}{}_{\nu}\, dx^{\nu} \otimes \frac{\partial}{\partial x^{\mu}} \quad \text{in local coordinates,}$$

is a graded derivation of degree one,

$$d_N \omega = N \lrcorner\, d\omega - d(N \lrcorner\, \omega) \quad \text{for any differential form } \omega$$

If N_1 and N_2 are two tensor fields of type $(1,1)$ with vanishing Nijenhuis brackets, $[N_i, N_j]_{\mathrm{FN}} = 0$, then $(\bigwedge(\mathcal{M}), d_{N_1}, d_{N_2})$ yields a bidifferential calculus:

$$d_{N_1}^2 = 0 \qquad d_{N_2}^2 = 0 \qquad d_{N_1}\, d_{N_2} = -d_{N_2}\, d_{N_1}$$

What do we get from our integrable equations in this case ?

M-H, *Nonlinearity* 2025:

- If $\dim(\mathcal{M}) = 2$, for any holomorphic function of a single complex variable, there is a non-autonomous deformation of the chiral model. (Here the linear system is also non-autonomous.)
- For special choices of the Nijenhuis tensor (in fixed coordinates), we recover the two potential versions of the sdYM equations, and also the corresponding hierarchy. Open: invariant characterization of the conditions.

Further interesting integrable systems in this FN case ?

Bidifferential calculus is a generalization of FN theory.

This year is Albert Nijenhuis' 100th anniversary. In 1958 he was invited speaker at the International Math. Conference in Edinburgh.

Concluding remarks

- Bidifferential calculus provides us with examples of integrable equations possessing a vectorial DT. It somehow represents the essence of a variety of examples, avoids repeated proofs, reveals a kind of universality.
- A vectorial DT is an improvement of the classical iterative DT.
    - More systematic approach to soliton solutions (no limit procedures necessary).
    - Regularity = invertibility of Ω. But convenient results about Sylvester equation with rhs of rank > 1 are still missing.
    - Superposition rule for solutions with same seed.
- We can let $N \to \infty$ (under suitable technical conditions). The bidiff. bDT theorem also holds in an operator framework. (In particular, there is relevant work of C. Schiebold.)
- "Cauchy matrix approach" essentially corresponds to vectorial DT with trivial seed. Verified e.g. for corresponding work on Fokas-Lenells equation (Li, Liu & Zhang '25, M-H & Ye '25).

Thanks for your attention !