Ergodic operator inequalities for unimodular amenable groups through random walks

Ujan Chakraborty

Connections between commutative and non-commutative harmonic analysis, ICMS

University of Glasgow
based on a joint work with Runlian Xia and Joachim Zacharias

4th September, 2025

Timeline for the development of Ergodic Theorems

- Pointwise ergodic theorems for L_p spaces:
  - For integers: Birkhoff, 1931
  - For amenable groups: Lindenstrauss, 2001

Timeline for the development of Ergodic Theorems

- Pointwise ergodic theorems for L_p spaces:
  - For integers: Birkhoff, 1931
  - For amenable groups: Lindenstrauss, 2001
- Counterparts for noncommutative L_p spaces:
  - For integers: Junge and Xu, 2005
  - For amenable groups: Cadilhac and Wang, 2022

Ergodic Theorems for L_p spaces

- $(X, \mathcal{B}, \mu)$: a probability space; T: a p.m.p. transformation on X.
- $M_n f = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$: ergodic average for meas. functions f on X.
- P_T: the projection onto the closed subspace of T-invariant functions.

Theorem (Birkhoff Ergodic Theorem)

For any $f \in L_1(X)$,

$$M_n f \to P_T f, \ \textit{a.e.} \ .$$

Ergodic Theorems for L_p spaces

- $(X, \mathcal{B}, \mu)$: a probability space; T: a p.m.p. transformation on X.
- $M_n f = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$: ergodic average for meas. functions f on X.
- P_T: the projection onto the closed subspace of T-invariant functions.

Theorem (Birkhoff Ergodic Theorem)

For any $f \in L_1(X)$,

$$M_n f \to P_T f, \text{ a.e. .}$$

- (G, λ): l.c.s.c. amenable group; α: p.m.p. action of G on X

Theorem (Lindenstrauss '01)

Given $(F_n)_{n\in\mathbb{N}}$ a tempered Følner sequence of G ($\lambda((\cup_{1\leqslant i<n} F_i)^{-1} F_n) \leqslant C\lambda(F_n)$, then for any $n \in \mathbb{N}$) for any $f \in L_1(X)$,

$$\frac{1}{\lambda(F_n)} \int_{F_n} \alpha_g(f) d\lambda(g) \to P(f), \text{ a.e. .}$$

Ergodic Theorems for L_p spaces

- $(X, \mathcal{B}, \mu)$: a probability space; T: a p.m.p. transformation on X.
- $M_n f = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$: ergodic average for meas. functions f on X.
- P_T: the projection onto the closed subspace of T-invariant functions.

Theorem (Birkhoff Ergodic Theorem)

For any $f \in L_1(X)$,

$$M_n f \to P_T f, \text{ a.e. .}$$

- (G, λ): l.c.s.c. amenable group; α: p.m.p. action of G on X

Theorem (Lindenstrauss '01)

Given $(F_n)_{n\in\mathbb{N}}$ a tempered Følner sequence of G ($\lambda((\cup_{1\leqslant i<n} F_i)^{-1} F_n) \leqslant C\lambda(F_n)$, then for any $n \in \mathbb{N}$) for any $f \in L_1(X)$,

$$\frac{1}{\lambda(F_n)} \int_{F_n} \alpha_g(f) d\lambda(g) \to P(f), \text{ a.e. .}$$

- X can also be σ-finite, also true for $L_p, p \in [1, \infty)$.

Maximal Ergodic Inequalities for NC L_p spaces

Theorem (Lance '76 (finite), Yeadon '77 (semifinite))

Let $x \in L_1^+(\mathcal{M},\tau)$, T a τ-preserving automorphism. Then for any $\alpha > 0$, there is $e \in \mathcal{P}(\mathcal{M})$ s.t.

$$\sup_{n\geqslant 1} \|eM_n(T)(x)e\|_\infty \leqslant \alpha \text{ and } \alpha\tau(1-e) \leqslant \|x\|_1.$$

Maximal Ergodic Inequalities for NC L_p spaces

Theorem (Lance '76 (finite), Yeadon '77 (semifinite))

Let $x \in L_1^+(\mathcal{M},\tau)$, T a τ-preserving automorphism. Then for any $\alpha > 0$, there is $e \in \mathcal{P}(\mathcal{M})$ s.t.

$$\sup_{n\geqslant 1} \|eM_n(T)(x)e\|_\infty \leqslant \alpha \text{ and } \alpha\tau(1-e) \leqslant \|x\|_1.$$

Theorem (Junge-Xu '06)

Let $(\mathcal{M},\tau)$ a semifinite vNa, T a positive linear map that is a contraction in norm and in trace. Then for any $p \in (1,\infty)$, we have

$$\| \sup\nolimits_n^+ M_n(T)(x)\|_p \leqslant C_p\|x\|_p,\ \forall x \in L_p(\mathcal{M}),$$

where $C_p \leqslant C\frac{p^2}{(1-p)^2}$, and this is the optimal order as $p \to 1$.

Individual Ergodic Theorems for NC L_p spaces

Definition

Let $\mathcal{M}$ be a vNa equipped with a n.s.f. trace τ. Let x_n, $x \in L_0(\mathcal{M})$.

- $x_n \longrightarrow x$ **bilaterally almost uniformly (b.a.u.)** if $\forall \epsilon > 0, \exists e \in \mathcal{P}(\mathcal{M})$ projection such that $\tau(1_{\mathcal{M}} - e) \leqslant \epsilon$, and $\lim_{n\to\infty} \|e(x_n - x)e\|_\infty = 0$.
- $x_n \longrightarrow x$ **almost uniformly (a.u.)** if $\forall \epsilon > 0, \exists e \in \mathcal{P}(\mathcal{M})$ projection such that $\tau(1_{\mathcal{M}} - e) \leqslant \epsilon$, and $\lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0$.

Individual Ergodic Theorems for NC L_p spaces

Definition

Let $\mathcal{M}$ be a vNa equipped with a n.s.f. trace τ. Let x_n, $x \in L_0(\mathcal{M})$.

- $x_n \longrightarrow x$ **bilaterally almost uniformly (b.a.u.)** if $\forall \epsilon > 0, \exists e \in \mathcal{P}(\mathcal{M})$ projection such that $\tau(1_{\mathcal{M}} - e) \leqslant \epsilon$, and $\lim_{n\to\infty} \|e(x_n - x)e\|_\infty = 0$.
- $x_n \longrightarrow x$ **almost uniformly (a.u.)** if $\forall \epsilon > 0, \exists e \in \mathcal{P}(\mathcal{M})$ projection such that $\tau(1_{\mathcal{M}} - e) \leqslant \epsilon$, and $\lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0$.

Theorem (Junge-Xu '06)

Let $x \in L_p(\mathcal{M})$. Then $M_n(x) \longrightarrow P(x)$ b.a.u. for $p \in [1, 2)$ and a.u. for $p \in [2, \infty)$.

Individual Ergodic Theorems for NC L_p spaces

Definition

Let $\mathcal{M}$ be a vNa equipped with a n.s.f. trace τ. Let $x_n, x \in L_0(\mathcal{M})$.

- $x_n \longrightarrow x$ **bilaterally almost uniformly (b.a.u.)** if $\forall \epsilon > 0, \exists e \in \mathcal{P}(\mathcal{M})$ projection such that $\tau(1_{\mathcal{M}} - e) \leqslant \epsilon$, and $\lim_{n\to\infty} \|e(x_n - x)e\|_\infty = 0$.
- $x_n \longrightarrow x$ **almost uniformly (a.u.)** if $\forall \epsilon > 0, \exists e \in \mathcal{P}(\mathcal{M})$ projection such that $\tau(1_{\mathcal{M}} - e) \leqslant \epsilon$, and $\lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0$.

Theorem (Junge-Xu '06)

Let $x \in L_p(\mathcal{M})$. Then $M_n(x) \longrightarrow P(x)$ b.a.u. for $p \in [1, 2)$ and a.u. for $p \in [2, \infty)$.

Theorem (Cadilhac-Wang '22)

Let G be a l.c.s.c. amenable group and (F_n) be a "regular filtered" Følner sequence on G. Then for $x \in L_p(\mathcal{M})$, the averages along F_n, $A_n(x) \longrightarrow P(x)$ b.a.u. for $p \in [1, 2)$ and a.u. for $p \in [2, \infty)$.

General background on unimodular amenable groups

⋆ Reiter's condition: A sequence of probability measures $(\mu_n)_n$ in $(L_1(G,\lambda))_1^+$ is called **(left) asymptotic invariant** if $\lim_{n\to\infty} \|\delta_g * \mu_n - \mu_n\|_1 = 0$

General background on unimodular amenable groups

⋆ Reiter's condition: A sequence of probability measures $(\mu_n)_n$ in $(L_1(G,\lambda))_1^+$ is called **(left) asymptotic invariant** if $\lim_{n\to\infty} \|\delta_g * \mu_n - \mu_n\|_1 = 0$

⋆ Følner condition: A sequence of measurable subsets of G with finite measure, $(F_n)_n$, is called a **(left) Følner sequence** if $\forall K \subset G$ compact, $\lim_{n\to\infty} \frac{\lambda(KF_n \triangle F_n)}{\lambda(F_n)} = 0$.

General background on unimodular amenable groups

⋆ Reiter's condition: A sequence of probability measures $(\mu_n)_n$ in $(L_1(G,\lambda))_1^+$ is called **(left) asymptotic invariant** if $\lim_{n\to\infty} \|\delta_g * \mu_n - \mu_n\|_1 = 0$

⋆ Følner condition: A sequence of measurable subsets of G with finite measure, $(F_n)_n$, is called a **(left) Følner sequence** if $\forall K \subset G$ compact, $\lim_{n\to\infty} \frac{\lambda(KF_n \triangle F_n)}{\lambda(F_n)} = 0$.

Definition

A l.c.s.c. group is called **amenable** if it satisfies either of the two (equivalent) conditions above (Reiter's / Følner's).

General background on unimodular amenable groups

⋆ Reiter's condition: A sequence of probability measures $(\mu_n)_n$ in $(L_1(G,\lambda))_1^+$ is called **(left) asymptotic invariant** if $\lim_{n\to\infty} \|\delta_g * \mu_n - \mu_n\|_1 = 0$

⋆ Følner condition: A sequence of measurable subsets of G with finite measure, $(F_n)_n$, is called a **(left) Følner sequence** if $\forall K \subset G$ compact, $\lim_{n\to\infty} \frac{\lambda(KF_n \triangle F_n)}{\lambda(F_n)} = 0$.

Definition

A l.c.s.c. group is called **amenable** if it satisfies either of the two (equivalent) conditions above (Reiter's / Følner's).

⋆ G is **unimodular** iff the $\lambda(E) = \lambda(E^{-1})$ for all measurable sets $E \subseteq G$, or equivalently, the left and right Haar measures are agree.

General background on unimodular amenable groups

⋆ Reiter's condition: A sequence of probability measures $(\mu_n)_n$ in $(L_1(G,\lambda))_1^+$ is called **(left) asymptotic invariant** if $\lim_{n\to\infty} \|\delta_g * \mu_n - \mu_n\|_1 = 0$

⋆ Følner condition: A sequence of measurable subsets of G with finite measure, $(F_n)_n$, is called a **(left) Følner sequence** if $\forall K \subset G$ compact, $\lim_{n\to\infty} \frac{\lambda(KF_n \triangle F_n)}{\lambda(F_n)} = 0$.

Definition

A l.c.s.c. group is called **amenable** if it satisfies either of the two (equivalent) conditions above (Reiter's / Følner's).

⋆ G is **unimodular** iff the $\lambda(E) = \lambda(E^{-1})$ for all measurable sets $E \subseteq G$, or equivalently, the left and right Haar measures are agree.

⋆ An amenable group can have both-sided Følner sequences ($(F_n)_n$ such that $\lim_{n\to\infty} \frac{\lambda(KF_nK \triangle F_n)}{\lambda(F_n)} = 0$) iff it is unimodular.

Dominating the Ergodic Average

Agenda: For a l.c.s.c. group G with Haar measure λ, a semifinite vNa $(\mathcal{M}, \tau)$, and a $w^\star$-continuous trace-preserving action $\alpha : (G, \lambda) \curvearrowright (\mathcal{M}, \tau)$, try to find:

- a suitable sequence of sets averaging F_n,
- a constant $C > 0$,
- a strictly increasing function $N : \mathbb{N} \to \mathbb{N}$, and
- a positive τ-preserving contraction T on $\mathcal{M}$,

such that for any positive x in $L_p^+(\mathcal{M})$ and any n,

$$\frac{1}{\lambda(F_n)} \int_{F_n} \alpha_g(x) d\lambda(g) \leqslant C \frac{1}{N(n)} \sum_{j=0}^{N(n)-1} T^j(x), \qquad (\dagger)$$

From Operator Dominations to Majorisation of Measures

Construct T in the following fashion: for some suitable probability measure $\omega \in (L_1(G,\lambda))_1^+$, we define $T(x) = \int \alpha_g(x) d\omega(g)$. Then, one has $T^n(x) = \int \alpha_g(x) d\omega^{(n)}(g)$, where $\omega^{(n)}$ is the n^{th} convolution of ω. Rearrange (†)

$$\frac{1}{\lambda(F_n)} \int_{F_n} \alpha_g(x) d\lambda(g) \leqslant C \frac{1}{N(n)} \sum_{j=0}^{N(n)-1} T^j(x)$$

to read

$$\int \alpha_g(x) \left(\frac{C}{N(n)} \sum_{j=0}^{N(n)-1} \frac{d\omega^{(j)}}{d\lambda}(g) - \frac{\chi_{F_n}(g)}{\lambda(F_n)} \right) d\lambda(g) \geqslant 0.$$

which is true for all vNa'a $(\mathcal{M},\tau)$ and all actions α *if and only if*

$$\frac{1}{N(n)} \sum_{j=0}^{N(n)-1} \frac{d\omega^{(j)}}{d\lambda}(g) \geqslant C^{-1} \frac{\chi_{F_n}(g)}{\lambda(F_n)}, \text{ for almost any } g \in G. \qquad \text{(MA)}$$

Majorisation of Measures

New Agenda: Given a l.c.s.c. group G with Haar measure λ, try to find:

- a suitable sequence of sets averaging F_n,
- a constant $C > 0$,
- a strictly increasing function $N : \mathbb{N} \to \mathbb{N}$, and
- a probability measure ω in $(L_1(G, \lambda))_1^+$,

such that for any n and for almost every $g \in F_n$,

$$\frac{\lambda(F_n)}{N(n)} \sum_{j=0}^{N(n)-1} \frac{d\omega^{(j)}}{d\lambda}(g) \geqslant C > 0.$$

Majorisation of Measures

New Agenda: Given a l.c.s.c. group G with Haar measure λ, try to find:

- a suitable sequence of sets averaging F_n,
- a constant $C > 0$,
- a strictly increasing function $N : \mathbb{N} \to \mathbb{N}$, and
- a probability measure ω in $(L_1(G, \lambda))_1^+$,

such that for any n and for almost every $g \in F_n$,

$$\frac{\lambda(F_n)}{N(n)} \sum_{j=0}^{N(n)-1} \frac{d\omega^{(j)}}{d\lambda}(g) \geqslant C > 0.$$

Question: Which groups allow that such F_n, N, and ω exist, and how do we find them?

Groups of Polynomial Growth:

Groups of polynomial growth (Hong-Liao-Wang '18): Consider the word metric w.r.t. some compact symmetric generating set, then the following choice is good enough:

- $F_n = B_n$ (ball of radius n centred at e),
- $N(n) = 2n^2$,
- $\frac{d\omega}{d\lambda}(g) = \frac{\chi_{B_1}(g)}{\lambda(B_1)}$.

Using a Gaussian estimate (Hebisch & Saloff-Coste '93),

$$\frac{d\omega^{(j)}}{d\lambda}(g) \geqslant \frac{ce^{-\frac{d(e,g)^2}{j}}}{\lambda\left(B_{\sqrt{j}}\right)}, \quad \forall g \in B_j,$$

one gets (MA) to hold.

Groups of Polynomial Growth:

Groups of polynomial growth (Hong-Liao-Wang '18): Consider the word metric w.r.t. some compact symmetric generating set, then the following choice is good enough:

- $F_n = B_n$ (ball of radius n centred at e),
- $N(n) = 2n^2$,
- $\frac{d\omega}{d\lambda}(g) = \frac{\chi_{B_1}(g)}{\lambda(B_1)}$.

Using a Gaussian estimate (Hebisch & Saloff-Coste '93),

$$\frac{d\omega^{(j)}}{d\lambda}(g) \geqslant \frac{ce^{-\frac{d(e,g)^2}{j}}}{\lambda\left(B_{\sqrt{j}}\right)}, \quad \forall g \in B_j,$$

one gets (MA) to hold.

Question

For amenable groups with faster growth, when we take F_n to be a Følner sequence satisfying some growth condition, what would be the choice of ω?

Going beyond polynomial growth

Motivated by a construction of Kaimanovich and Vershik '83, we define:

$$\frac{d\omega}{d\lambda} = \sum_{n\in\mathbb{N}} t_n \frac{\chi_{E_n}}{\lambda(E_n)}.$$

Consequently,

$$\frac{d\omega^{(j)}}{d\lambda} = \sum_{i_1,\ldots,i_j\in\mathbb{N}} t_1\ldots t_j \left(\frac{\chi_{E_{i_1}}}{\lambda(E_{i_1})}\right) \star\cdots\star \left(\frac{\chi_{E_{i_j}}}{\lambda(E_{i_j})}\right).$$

Aim: Find conditions on E_n and F_n to make

$$\frac{1}{N(n)} \sum_{j=0}^{N(n)-1} \frac{d\omega^{(j)}}{d\lambda}(g) \geqslant C\frac{\chi_{F_n}(g)}{\lambda(F_n)}, \text{ for any } n\in\mathbb{N}, \qquad \text{(MA)}$$

that is,

$$\frac{1}{N(n)} \sum_{j=0}^{N(n)-1} \sum_{i_1,\ldots,i_j\in\mathbb{N}} t_1\ldots t_j \left(\frac{\chi_{E_{i_1}}}{\lambda(E_{i_1})}\right) \star\cdots\star \left(\frac{\chi_{E_{i_j}}}{\lambda(E_{i_j})}\right) \geqslant C\frac{\chi_{F_n}(g)}{\lambda(F_n)}. \quad \text{(MA)}$$

Lemma (A Wee Observation)

For any compact subsets $E, F \subset G$, the following are equivalent:

1. $\left(\left(\frac{\chi_E}{\lambda(E)}\right) \star \chi_F\right)(g) = 1$;
2. $E^{-1}g \subset F$.

Similarly, the following are also equivalent:

1. $\left(\chi_F \star \left(\frac{\chi_E}{\lambda(E)}\right)\right)(g) = 1$;
2. $gE^{-1} \subset F$.

Lemma (A Wee Observation)

For any compact subsets $E, F \subset G$, the following are equivalent:

1. $\left(\left(\frac{\chi_E}{\lambda(E)}\right) \star \chi_F\right)(g) = 1$;
2. $E^{-1}g \subset F$.

Similarly, the following are also equivalent:

1. $\left(\chi_F \star \left(\frac{\chi_E}{\lambda(E)}\right)\right)(g) = 1$;
2. $gE^{-1} \subset F$.

Using this, we are motivated that in the product $\left(\frac{\chi_{E_{i_1}}}{\lambda(E_{i_1})}\right) \star \cdots \star \left(\frac{\chi_{E_{i_j}}}{\lambda(E_{i_j})}\right)$, if a certain i_k (viz. the largest one) is such that E_{i_k} is contains the translates of F_n from both sides by all the other factors, then for evaluation on F_n, we may replace this product simply by $\frac{\chi_{E_{i_k}}}{\lambda(E_{i_k})}$.

Our Theorem

Theorem (C.-Xia-Zacharias)

If the following assumptions hold, then we will have majorisation (MA).

1. *For any* $n \in \mathbb{N}$, $\left(E_{n-1}^{N(n)-2}\right)^{-1} F_n \left(E_{n-1}^{N(n)-2}\right)^{-1} \subseteq E_n$.
2. $\lim_{n\to\infty} \frac{r_{n+1}}{r_n} \in [0,1)$, *where* $r_n = \sum_{j=n}^{\infty} t_j$.
3. $0 < \lim_{n\to\infty} N(n) r_n < \infty$.
4. $\lim_{n\to\infty} \left(\frac{\lambda(F_n)}{\lambda(E_n)}\right) > 0$.

In particular, when G *is a l.c.s.c. unimodular amenable group, such* E_n *and* F_n *exist.*

Choosing $t_n = \frac{1}{2^n}$ and $N(n) = 2^n$, we have that for all $g \in F_n$,

$$\lim_{n\to\infty} \frac{\lambda(F_n)}{N(n)} \sum_{j=0}^{N(n)-1} \frac{d\omega^{(j)}}{d\lambda}(g) \geqslant \frac{1}{4}\left(1 - \frac{3}{e^2}\right)\left(\lim_{n\to\infty} \frac{\lambda(F_n)}{\lambda(E_n)}\right).$$

The construction of E_n and F_n for amenable groups

Step 1: $E_1 = F_1$.

Step 2: For $n \geqslant 2$, choose F_n to be a Følner set that is $\left(\left(E_{n-1}^{N(n)-2}\right)^{-1}, \epsilon_n\right)$-bi-invariant, with $\lim_{n\to\infty} \epsilon_n = 0$.

Step 3: For $n \geqslant 2$, define $E_n = \left(E_{n-1}^{N(n)-2}\right)^{-1} F_n \left(E_{n-1}^{N(n)-2}\right)^{-1}$.

Observation:

If F_n's contain e and are symmetric, then

- $\frac{\lambda(F_n)}{\lambda(E_n)} = \frac{1}{1+\frac{\lambda(E_n \setminus F_n)}{\lambda(F_n)}} \longrightarrow \frac{1}{1+0} = 1 > 0.$
- Every both-sided Følner sequence admits a subsequence satisfying the above assumption.

Applications to ergodic theorem

Corollary

Let G be a discrete amenable group and $(F_n)_n$ be a Følner sequence satisfying the condition in the previous slide. The operators defined by

$$A_n(x) = \frac{1}{\lambda(F_n)} \int_{F_n} \alpha_g x \, d\lambda(g)$$

is weak type $(1,1)$ and strong type (p,p) for $p \in (1,\infty)$.

Corollary

For any $x \in L_p(\mathcal{M})$, $A_n x$ converges to Px b.a.u. for $p \in [1,2)$; and a.u. for $p \in [2,\infty)$.

Examples / Comparison of averaging Følner sets:

⋆**Groups of Polynomial growth**

- Lindenstrauss: $F_n = B_n$,
- Our sets: $F_n = B_{2^{n^2}}$.

⋆**Lamplighter group** (Exponential growth)
$\mathbb{Z} \ltimes \oplus_{i\in\mathbb{Z}}\mathbb{Z}_2$: $(i,a)\cdot(j,b) = (i+j, \sigma_j a + b)$, where $\sigma : \oplus_{i\in\mathbb{Z}}\mathbb{Z}_2 \to \oplus_{i\in\mathbb{Z}}\mathbb{Z}_2$, $(\sigma a)_n = a_{n-1}$. (Left) Følner sequences given by $F_n^0 = \{(t, \sum_{k=0}^{k=n} \alpha_k e_k) : t \in [0,n], \alpha_k \in \{0,1\}\}$ and $|F_n^0| = (n+1)2^{n+1}$. Both-sided Følner sequences given by $F_n = F_n^0 (F_n^0)^{-1}$, $|F_n| = 2^n(4n^2+4n+2)$. Then,

- Lindenstrauss: A subsequence $F^0_{l(n)}$ where $l(n) = 2^{4l(n-1)}$,
- Our sets: A subsequence $F_{l(n)}$ where $l(n) = 2^{64n^2 l(n-1)}$,

Thank you!