Twisting asymptotically-flat spacetimes

Marc Geiller
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work with Pujian Mao and Antoine Vincenti, [arXiv:2509.xxxxx]

International Centre for Mathematical Sciences

AdS/CFT meets Carrollian & Celestial Holography
September 8-12 2025

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(part 2) (part 1)

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CNRS PHYSIQUE · Université de Lyon · ENS DE LYON · Lab PHYS ENS de LYON

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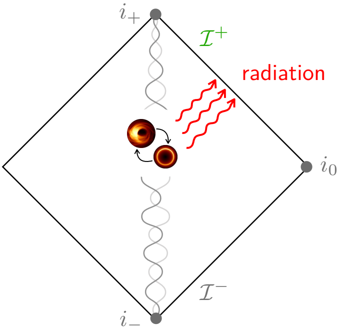

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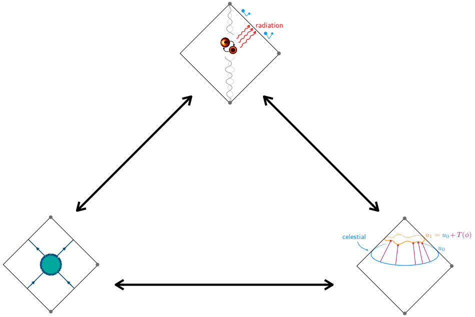

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  - memory effects [Blanchet, Christodoulou, Damour, Polnarev, Thorne, Zel'dovich]
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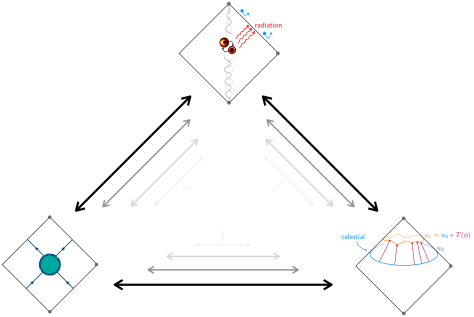

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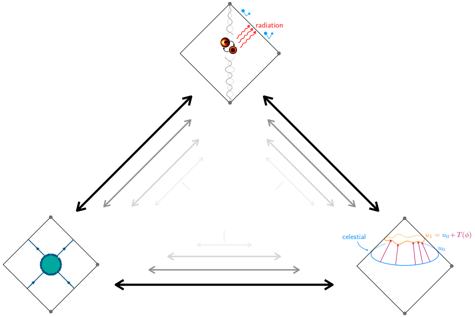

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- What is the classical (geometric, algebraic) structure underlying flat space holography?

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- Near $\mathcal{I}^+$, a lot of precious information can be extracted
  - asymptotic symmetries, transformation laws, charges and their algebra, ...
  - flux-balance laws, memory effects, ...
  - link with numerical relativity, waveforms, ...

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Generalizations and alternatives

Asymptotically-flat spacetimes

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- Within the Bondi formalism, one can choose different radial coordinates, e.g.
  - Bondi–Sachs gauge: $\det(\gamma_{ab}) = r^4 \det(q_{ab})$ and r is the areal distance
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  - arbitrary q_{ab} (to go from BMS to GBMS or BMSW) [Barnich, Campiglia, Flanagan, Freidel, Laddha, Nichols, Oliveri, Pranzetti, Speziale, Trossaert, ...]
  - $\partial_u q_{ab} \neq 0$ (useful for Robinson–Trautman and (A)dS radiation) [Adami, Barnich, Ciambelli, Compère, Hoque, Kutluk, Pavizi, Petkou, Petropoulos, Seraj, Sheikh-Jabbari, Siampos, Taghiloo, ...]
  - $\log(r)$ terms and violations of peeling [Bieri, Chruściel, Christodoulou, Damour, MacCallum, Friedrich, Gajic, MG, Kehrberger, Kroon, Laddha, Masaood, Singleton, Winicour, Zwikel, ...]
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- The mapping between gauges is subtle due to the field-dependency of the diffeomorphism, e.g. [Compère, Long] [Ciambelli, MG]

$$\tilde{\xi}^\mu = J^\mu_\alpha \big(\xi^\alpha + \delta_\xi x^\alpha\big)$$

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- When using the "Bondi tetrad" $(\ell, n, m, \bar{m})$

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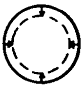
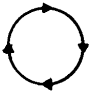
- Kerr is type D algebraically special, so with a PND aligned tetrad we have $\Psi_0 = \Psi_1 = 0$
- This motivates us to introduce the **twist**, i.e. $\mathbf{Im}(\boldsymbol{\rho}) \neq \mathbf{0}$
- This means merging Bondi with [Chandrasekhar] [Stephani, Kramer, MacCallum, Hoenselaers, Herlt]

$$\rho = m^\mu \bar{m}^\alpha \nabla_\alpha \ell_\mu$$

$$\sigma = m^\mu m^\alpha \nabla_\alpha \ell_\mu$$



Re ρ



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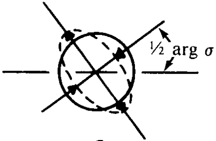

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- Solving the Einstein equations in NP form is extremely efficient, but obscures a few things
  - the nature of the gauge and no-log conditions, and the Bondi hierarchy of equations
  - the extension of the "familiar" Bondi expressions to $L\neq 0$

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- The upshot is

$$\begin{aligned}
\Psi_0 &= \frac{\Psi_0^0}{r^5} + \frac{\Psi_0^1}{r^6} + \mathcal{O}(r^{-7}) \\
\Psi_1 &= \frac{\Psi_1^0}{r^4} - \frac{\bar{\eth}\Psi_0^0 + 4i\Sigma\Psi_1^0}{r^5} + \mathcal{O}(r^{-6}) \\
\Psi_2 &= \frac{\Psi_2^0}{r^3} - \frac{\bar{\eth}\Psi_1^0 + 3i\Sigma\Psi_2^0}{r^4} + \mathcal{O}(r^{-5}) \\
\Psi_3 &= \frac{\Psi_3^0}{r^2} - \frac{\bar{\eth}\Psi_2^0 + 2i\Sigma\Psi_2^0}{r^3} + \mathcal{O}(r^{-4}) \\
\Psi_4 &= \frac{\Psi_4^0}{r} - \frac{\bar{\eth}\Psi_3^0 + i\Sigma\Psi_4^0}{r^2} + \mathcal{O}(r^{-3})
\end{aligned}$$

Twisting asymptotically-flat spacetimes

Solution space in NP form

- One can always solve the NP equations with $\kappa = \epsilon = \pi = 0$, so that $\ell^\mu \nabla_\mu e = 0$
- The upshot is

$$\rho = -\frac{1}{r} + \frac{i\Sigma}{r^2} + \frac{\Sigma^2 - \sigma_2\bar{\sigma}_2}{r^3} - \frac{i\Sigma\rho_3}{r^4} + \mathcal{O}(r^{-5})$$

$$\sigma = \frac{\sigma_2}{r^2} - \frac{2\sigma_2\rho_3 + \Psi_0^0}{2r^4} - \frac{\Psi_0^1}{3r^5} + \mathcal{O}(r^{-6})$$

$$\alpha = \frac{\alpha_1}{r} + \frac{\bar{\alpha}_1\bar{\sigma}_2 - i\Sigma\alpha_1}{r^2} - \frac{\alpha_1\rho_3}{r^3} + \mathcal{O}(r^{-4})$$

$$\beta = -\frac{\bar{\alpha}_1}{r} - \frac{\alpha_1\sigma_2 + i\Sigma\bar{\alpha}_1}{r^2} + \frac{2\bar{\alpha}_1\rho_3 - \Psi_1^0}{2r^3} + \mathcal{O}(r^{-4})$$

$$\tau = \frac{\tau_1}{r} - \frac{\sigma_2\bar{\tau}_1 + i\Sigma\tau_1}{r^2} - \frac{2\tau_1\rho_3 + \Psi_1^0}{2r^3} + \mathcal{O}(r^{-4})$$

$$\lambda = \frac{\lambda_1}{r} - \frac{\bar{\sigma}_2\mu_1 + i\Sigma\lambda_1}{r^2} + \mathcal{O}(r^{-3})$$

$$\mu = \frac{\mu_1}{r} - \frac{\sigma_2\lambda_1 - i\Sigma\mu_1 + \Psi_2^0}{r^2} + \mathcal{O}(r^{-3})$$

$$\gamma = \gamma_0 + \frac{\bar{\alpha}_1\bar{\tau}_1 - \alpha_1\tau_1}{r} + \frac{2\alpha_1\sigma_2\bar{\tau}_1 - 2\bar{\alpha}_1\bar{\sigma}_2\tau_1 + 2i\Sigma(\alpha_1\tau_1 + \bar{\alpha}_1\bar{\tau}_1) - \Psi_2^0}{2r^2} + \mathcal{O}(r^{-3})$$

$$\nu = \nu_0 - \frac{\lambda_1\tau_1 + \mu_1\bar{\tau}_1 + \Psi_3^0}{r} + \mathcal{O}(r^{-2})$$

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$$
\begin{aligned}
2i\Sigma &= \eth \bar{L} - \bar{\eth} L \\
\rho_3 &= \Sigma^2 - \sigma_2 \bar{\sigma}_2 \\
\alpha_1 &= -\frac{1}{2} D_a \bar{m}_1^a - \gamma_0 \bar{L} \\
\tau_1 &= -\big(\partial_u + 2\gamma_0\big) L \\
\lambda_1 &= \big(\partial_u + 2\gamma_0\big)\bar{\sigma}_2 - \big(\bar{\eth} - \bar{\tau}_1\big)\bar{\tau}_1 \\
\mu_1 &= 2i\Sigma\gamma_0 - \mu_R \\
\mu_R &= \eth\alpha_1 + \bar{\eth}\bar{\alpha}_1 \\
\gamma_0 &= \frac{1}{4}\partial_u \ln\sqrt{q} \\
\nu_0 &= 2\big(\bar{\eth} - \bar{\tau}_1\big)\gamma_0 \\
\Psi_3^0 &= -\eth\lambda_1 + \bar{\eth}\mu_1 + 2i\Sigma\nu_0 \\
\Psi_4^0 &= \big(\bar{\eth} - \bar{\tau}_1\big)\nu_0 - \big(\partial_u + 4\gamma_0\big)\lambda_1 \\
\Psi_2^0 - \bar{\Psi}_2^0 &= \bar{\lambda}_1\bar{\sigma}_2 - \lambda_1\sigma_2 + \bar{\eth}\omega_1 - \eth\bar{\omega}_1 + 2i\Sigma\big(U_0 - \mu_R\big)
\end{aligned}
$$

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\begin{aligned}
W &= 1-\frac{2\mathrm{Re}(L\bar\tau_1)}{r}+\frac{2\mathrm{Re}\big(L(\bar\sigma_2\tau_1-i\Sigma\bar\tau_1)\big)}{r^2}+\mathcal{O}(r^{-3})\\
U &= -2\gamma_0 r+\big(\bar{\eth}-\bar\tau_1\big)\tau_1+\bar\mu_1-i(\partial_u+2\gamma_0)\Sigma-\frac{\mathrm{Re}\big(\Psi_2^0+2\tau_1\bar\omega_1\big)}{r}+\mathcal{O}(r^{-2})\\
X^a &= -\frac{2\mathrm{Re}(m_1^a\bar\tau_1)}{r}+\frac{2\mathrm{Re}\big(m_1^a(\bar\sigma_2\tau_1-i\Sigma\bar\tau_1)\big)}{r^2}+\frac{\mathrm{Re}\big(m_1^a(\bar\Psi_1^0+6\rho_3\bar\tau_1)\big)}{3r^3}+\mathcal{O}(r^{-4})\\
Z &= \frac{L}{r}+\frac{i\Sigma L-\sigma_2\bar L}{r^2}-\frac{\rho_3 L}{r^3}+\mathcal{O}(r^{-4})\\
\omega &= \frac{\bar{\eth}\sigma_2-i\big(\eth-2\tau_1\big)\Sigma}{r}+\frac{2i\Sigma\omega_1-2\sigma_2\bar\omega_1-\Psi_1^0}{2r^2}+\mathcal{O}(r^{-3})\\
m^a &= \frac{m_1^a}{r}+\frac{i\Sigma m_1^a-\sigma_2\bar m_1^a}{r^2}-\frac{\rho_3 m_1^a}{r^3}+\frac{\Psi_0^0\bar m_1^a-6\rho_3 m_2^a}{6r^4}+\mathcal{O}(r^{-5})
\end{aligned}
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\begin{aligned}
(\partial_u + 2\gamma_0)\alpha_1 &= -(\bar{\eth} - \bar{\tau}_1)\gamma_0 \\
(\partial_u + 4\gamma_0)\mu_1 &= (\eth - \tau_1)\nu_0 \\
(\partial_u + 6\gamma_0)\Psi_3^0 &= (\eth - \tau_1)\Psi_4^0 \\
(\partial_u + 6\gamma_0)\Psi_0^0 &= (\eth - 4\tau_1)\Psi_1^0 + 3\sigma_2\Psi_2^0 \\
(\partial_u + 6\gamma_0)\Psi_1^0 &= (\eth - 3\tau_1)\Psi_2^0 + 2\sigma_2\Psi_3^0 \\
(\partial_u + 6\gamma_0)\Psi_2^0 &= (\eth - 2\tau_1)\Psi_3^0 + 1\sigma_2\Psi_4^0
\end{aligned}
$$

Twisting asymptotically-flat spacetimes

Solution space in NP form

- One can always solve the NP equations with $\kappa=\epsilon=\pi=0$, so that $\ell^\mu\nabla_\mu e=0$
- The key features are that
  - peeling is preserved
  - the twist enters $2i\Sigma=\text{Đ}\bar{L}-\bar{\text{Đ}}L=i(D^a+L^a\partial_u)\widetilde{L}_a$ and $\text{Im}(\rho)=\Sigma$
  - the $\eth$ operator is replaced by $\text{Đ}=\eth-2s\gamma_0L+L\partial_u$
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Solution space in metric form

- Even though $g_{\mu\nu}=e^i_\mu e^j_\nu\eta_{ij}$, mapping everything to metric form is painful yet instructive
- The gauge fixing is complete, and the gauge conditions are
  - $g_{rr}=0$
  - $\text{Re}(\epsilon)=0$, which implies the Newman–Unti gauge condition $g_{ur}=-1$
  - $\kappa=\bar{\kappa}=0$, which implies $\partial_rL_a=0$

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- Flux-balance laws in “familiar form”, e.g. when $\gamma_0=0=\partial_uL$

$$\partial_uE=-\frac{1}{4}N_{ab}N^{ab}+\frac{1}{2}D_a\mathcal{J}^a\qquad\qquad E:=M+\text{complicated}$$

Twisting asymptotically-flat spacetimes

Asymptotic symmetries

- Following the usual construction (i.e. preserving the gauge and the fall-offs) we find

$$\xi^u = \underbrace{T + uW}_{f} + L_a(\xi^a - Y^a) \qquad \xi^r = rW + \sum_{n=0}^{\infty} \frac{\xi^r_n(\Upsilon)}{r^n} \qquad \xi^a = Y^a + \frac{\Upsilon^a}{r} + \sum_{n=2}^{\infty} \frac{\xi^a_n(\Upsilon)}{r^n}$$

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- The solution space truncates as well!

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- Everything can easily be inferred by reducing the algebraically general solution space
- At the end of the day, the solution space is controlled by
  - time-dependent twist $L_a(u, x^b)$ and dyad (or celestial metric) $m^a(u, x^b)$
  - the mass $\mathrm{Re}(\Psi_2^0)$ subject to constraints and an evolution equation

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- One can easily compute the charges for any such algebraically special solution

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Supertranslated Schwarzschild

- Simple example: finitely supertranslated Schwarzschild obtained by $u \mapsto u - C(x^a)$

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r}\right)(du - dC)^2 - 2(du - dC)dr + r^2 d\Omega^2 \\
&= -\left(1 - \frac{2M}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2 \\
&\quad + \left[2\left(1 - \frac{2M}{r}\right)du + 2dr - \left(1 - \frac{2M}{r}\right)\partial_a C dx^a\right]\left(\partial_a C dx^a\right)
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- The metric can be written in isotropic coordinates [Compère, Long]
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- Simple example: finitely supertranslated Schwarzschild obtained by $u \mapsto u - C(x^a)$

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- NB: the tetrad can further be Lorentz transformed to the so-called Kinnersley tetrad (type D)

Conclusion and perspectives

What's new

- Complete NP and metric characterization of the algebraically general solution space with twist
- Natural reduction to the algebraically special solutions
- 3d version with $\Lambda \neq 0$ straightforward

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- Study the various extensions mentioned previously (e.g. 4d $\Lambda \neq 0$, logs, FLRW, matter, ...)
- Near-horizon construction in the algebraically general case, link with isolated horizons
- Detailed study of the charges in the algebraically special case
- Bringing radiative solutions from harmonic gauge to Bondi with twist (simpler than [BCFOS]?)
- Useful for numerical relativity?
- Coordinate-free statements *à la* Geroch
- Subleading symmetries and $w_{1+\infty}$
- Interplay between asymptotic symmetries and Killing–Stäckel/Yano tensors, e.g. for type D

$$K_{\mu\nu} = 2(r^2 + \Sigma^2)\ell_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}$$

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Thanks for your attention!