Integrable Systems

Non-Commutative (NC) Spaces

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18 June 2026 in Edinburgh

Sponsored by Asahipen Hikari Foundation
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& Toyoaki Foundation & ICMS & Heriot-Watt & Edinburgh U.

§1 Introduction

Non-Commutative (NC) Space :

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad \cdots (*) \qquad (\theta^{\mu\nu} : \text{NC parameter, real})$$

Three merits :

① Resolution of singularity

② description of bkg magnetic fields

③ easy treatments

§1 Introduction

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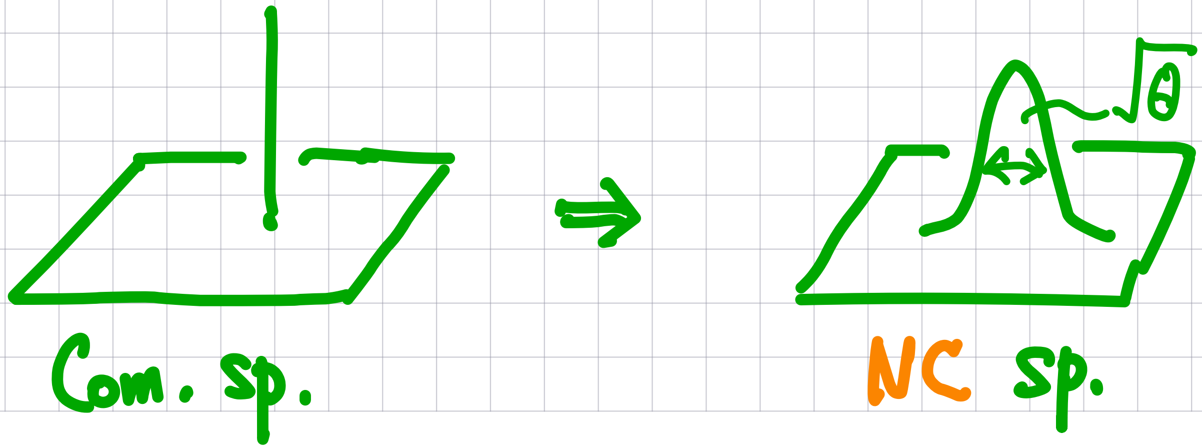
$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad \cdots (*) \qquad (\theta^{\mu\nu}: \text{NC parameter, real})$$

Three merits :

① Resolution of singularity

② (*) looks like CCR in QM ⇝ ("space-space uncertaincy")

⇝



§1 Introduction

Non-Commutative (NC) Space :

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad \cdots (*)$$

($\theta^{\mu\nu}$: NC parameter, real)

Three merits :

① Resolution of singularity

② description of bkg magnetic fields

③ easy treatments

due to $\begin{cases} \text{the merit ①} \\ \text{Quasideterminant descriptions} \end{cases}$

As for NC extension, we focus

- NC instantons (to explain ① & ②)
- NC KdV eqs. & Quasideterminants (③)

Finally, I would discuss unified theories of integrable systems in gauge theories:

- 4dim ASD Yang-Mills (YM) eq.
- 4dim Chern-Simons (CS) theory

(with relation to KP eq.)

Plan of Talk

§1 Introduction (5 min)

§2 NC integrable systems (7 min)

§3 ADHM Construction of instantons (13 min)

§4 NC KdV eq & Quasideterminants (10 min)

§5 ASDYM revisited (v.s. KP) (10 min)

~ Towards Unification of Integrable Systems

§2. NC Integrable Systems

ASDYM eq. (in 4 dim., G: gauge gp.)

$\mathbb{R}^4$

$$*F_{\mu\nu} = -F_{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Hodge dual　　　field strength　　　gauge field　　　commutator

$\mu, \nu \in \{1, 2, 3, 4\}$

$$\Leftrightarrow \quad F_{12} = -F_{34}, \quad F_{13} = -F_{42}, \quad F_{14} = -F_{23}$$

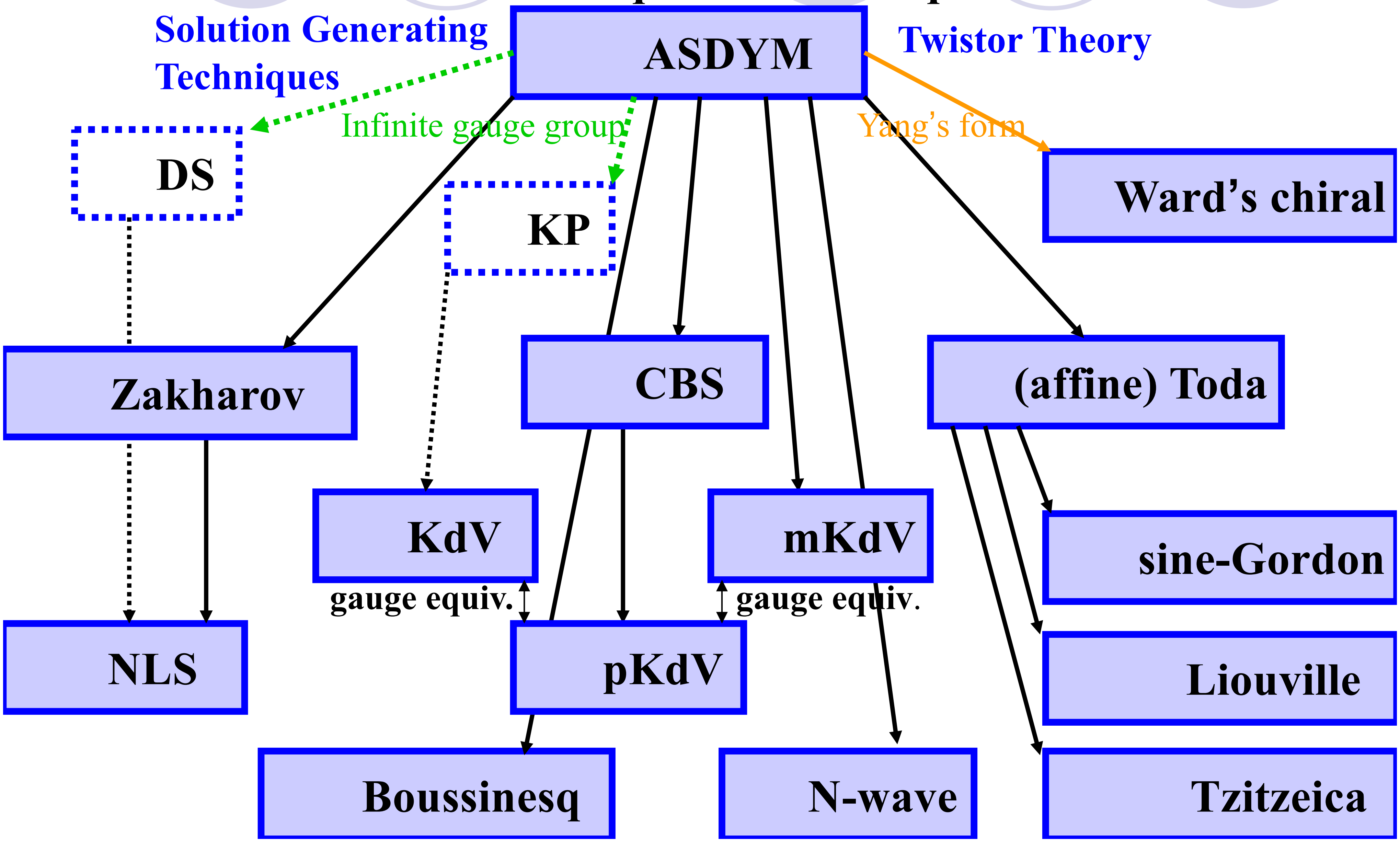
Note A_μ, $F_{\mu\nu}$ take value in $\mathfrak{g}$ (Lie alg. of G)

e.g. $G = U(N)$, A_μ, $F_{\mu\nu}$ are $N \times N$ anti-hermitian.

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

Summarized in the book of Mason&Woodhouse

ASDYM eq. is a master eq. !



Reduction to KdV from ASDYM $(G = SL(2,\mathbb{C}))$

ASDYM : $F_{zw} = 0,\ F_{\tilde{z}\tilde{w}} = 0,\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0$

① $\partial_w - \partial_{\tilde{w}} = 0,\ \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ \frac{u}{2} & 0 \end{pmatrix},\ A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_w = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}$

$A_z = \frac{1}{4}\begin{pmatrix} u' & -2u \\ u'' + 2u^2 & -u' \end{pmatrix}$ $\quad u = u(z, \underset{\sim}{x}), \quad \underbrace{x = w + \tilde{w}}$

$u' = \partial_x u$

$$u_z - \frac{1}{4}u_{xxx} - \frac{3}{2}u u_x = 0 \quad : \text{KdV eq.!}$$

z $\overset{\text{“}}{t}$ $\quad (t, x)$ are real $\Rightarrow$ not $(++++)$ but $(++--)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (E) \qquad\qquad\qquad (U)$

Reduction to NLS from ASDYM $(G = SL(2,\mathbb{C}))$

ASDYM : $F_{zw} = 0,\ F_{\tilde{z}\tilde{w}} = 0,\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0$

$\downarrow$

① $\partial_w - \partial_{\tilde{w}} = 0,\ \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = O,\ A_{\tilde{z}} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A_w = \begin{pmatrix} 0 & \psi \\ \varepsilon\bar{\psi} & 0 \end{pmatrix}$ $\quad \varepsilon = \pm 1$

$A_z = i\varepsilon \begin{pmatrix} -|\psi|^2 & -\varepsilon\psi' \\ \bar{\psi}' & |\psi|^2 \end{pmatrix}$ $\quad \psi = \psi(z, \underset{\sim}{x}), \ x = \underset{\sim\sim\sim}{w + \tilde{w}}$

$\psi' = \partial_x \psi$

$i\psi_z - \psi_{xx} + 2\varepsilon\psi^2\bar{\psi} = 0$: **NLS eq.!**

$\overset{\text{"}}{t}$ $\quad$ (t, x) are real $\Rightarrow$ not $(++++)$ but $(++--)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (E) \qquad\qquad\qquad (U)$

(Q) How to obtain NC theory from a com. theory?

(A) NC field eqs. are defined from commutative field eqs. by replacement of all products with Moyal products

$$f(x) \star g(x) := f(x)\, e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\,\overrightarrow{\partial_\nu}}\, g(x)$$

c-number

$$= f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f\cdot\partial_\nu g + O(\theta^2)$$

$$[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = x^\mu\cdot x^\nu + \frac{i}{2}\theta^{\mu\nu} - \left(x^\nu\cdot x^\mu + \frac{i}{2}\theta^{\nu\mu}\right) = i\theta^{\mu\nu}$$

§2. NC Integrable Systems

NC ASDYM eq. (in 4 dim., G: gauge gp.)

$$*F^\star_{\mu\nu} = -F^\star_{\mu\nu} \qquad F^\star_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\star$$

must be kept even for $G = U(1)$

$$\mu, \nu \in \{1, 2, 3, 4\}$$

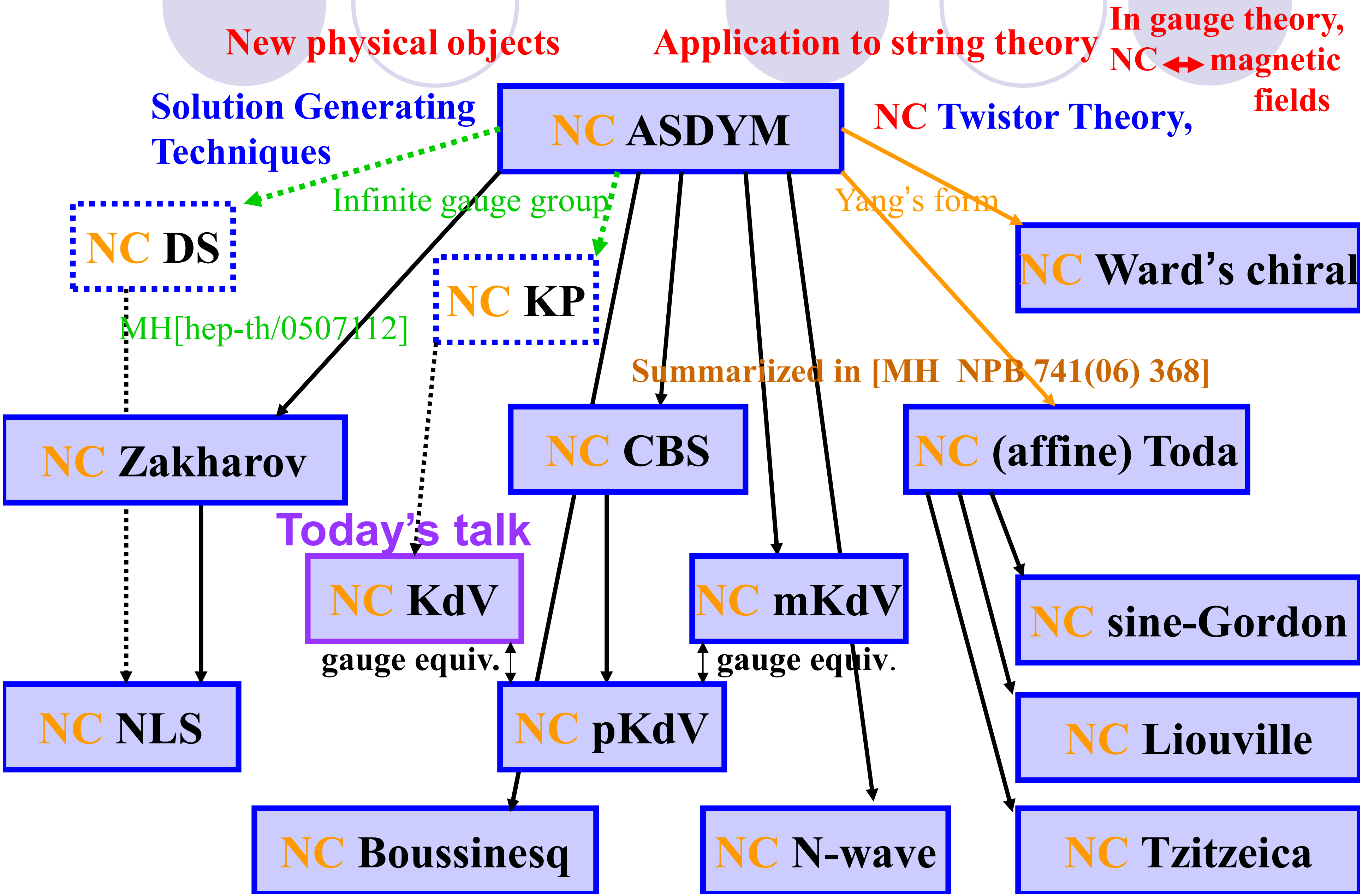
$$\Leftrightarrow \quad F^\star_{12} = -F^\star_{34}, \quad F^\star_{13} = -F^\star_{42}, \quad F^\star_{14} = -F^\star_{23}$$

Note A_μ, $F_{\mu\nu}$ take value in $\mathfrak{g}$ (Lie alg. of G)

e.g. $G = U(N)$, A_μ, $F_{\mu\nu}$ are $N \times N$ anti-hermitian.

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

MH &K.Toda, PLA316 (03)77 [hepth/0211148]



Reduction to KdV from NC ASDYM $(G = SL(2,\mathbb{C}))$

NC ASDYM : $F^{\star}_{zw} = 0, \ F^{\star}_{\tilde{z}\tilde{w}} = 0, \ F^{\star}_{z\tilde{z}} - F^{\star}_{w\tilde{w}} = 0$

① $\partial_w - \partial_{\tilde{w}} = 0, \ \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ \frac{u}{2} & 0 \end{pmatrix}, \ A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ A_w = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}$

$A_z = \frac{1}{4}\begin{pmatrix} u' & -2u \\ u'' + 2u^2 & -u' \end{pmatrix}$ $\quad u = u(z, \underset{\sim}{x}), \ \underbrace{x}_{} = w + \tilde{w}, \quad u' = \partial_x u$

$$\underset{\substack{\text{III} \\ t}}{u_z} - \frac{1}{4}u_{xxx} - \frac{3}{4}\underbrace{(u \star u_x + u_x \star u)}_{\text{Symmetric}} = 0 \quad : \text{NC KdV eq. !}$$

Reduction to NC NLS from NC ASDYM $(G=GL(2,\mathbb{C}))$

NCASDYM : $F^\star_{zw}=0,\ F^\star_{\tilde{z}\tilde{w}}=0,\ F^\star_{z\tilde{z}}-F^\star_{w\tilde{w}}=0$

① $\partial_w-\partial_{\tilde{w}}=0,\ \partial_{\tilde{z}}=0$ (dim. reduction)

② $A_{\tilde{w}}=0,\ A_{\tilde{z}}=\frac{i}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ A_w=\begin{pmatrix}0&\psi\\ \varepsilon\bar{\psi}&0\end{pmatrix}$ $\quad\varepsilon=\pm1$

$A_z=i\varepsilon\begin{pmatrix}-\bar{\psi}\star\psi & -\varepsilon\psi'\\ \bar{\psi}' & \psi\star\bar{\psi}\end{pmatrix}$ $\quad\psi=\psi(z,x^{w+\tilde{w}})$

$\psi'=\partial_x\psi$

$i\psi_{\underset{t}{z}}-\psi_{xx}+2\varepsilon\psi\star\bar{\psi}\star\psi=0$: NC NLS eq.!

★ trace part (U(1) part in gauge gp) is crucial!

- NC Bogomol'nyi eq. $B_i = D_i \Phi$ (products are all Moyal)
- NC KP eq. $u_t = u_{xxx} + u_x u + u u_x + \partial_x^{-1} u_{yy} - [u, \partial_x^{-1} u_y]$
- NC Zakhalov system $(\varepsilon = \pm 1)$

$$i\psi_t = \psi_{xy} - \varepsilon \psi \partial_x^{-1} \partial_y (\bar{\psi} \psi) - \varepsilon \partial_x^{-1} \partial_y (\psi \bar{\psi}) \psi$$

- NC Bogoyavlenskii-Calogero-Schiff eq.

$$u_t = u_{xxy} + u u_y + u_y u + u_y \partial_x^{-1} u_y + (\partial_x^{-1} u_y) u_y + \partial_x^{-1} [u, \partial_x^{-1} [u, \partial_x^{-1} u_y]]$$

- NC Davey-Stewartson eq.

$$\begin{cases} 2\kappa q_t = (q_{xx} + \kappa^2 q_{yy}) + R_1 q - q R_2 \\ 2\kappa r_t = -(r_{xx} + \kappa^2 r_{yy}) + R_2 r - r R_1 \end{cases} \quad \text{w/} \quad \begin{cases} R_{1,z} = -(qr)_{\bar{z}} \\ R_{2,\bar{z}} = (rq)_z \end{cases}$$

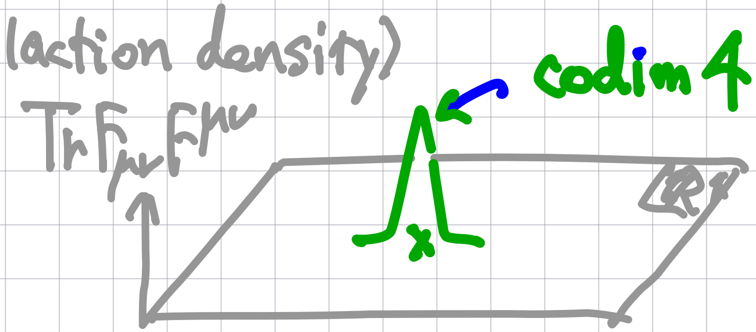
- NC KdV eq. $u_t = u_{xxx} + u u_x + u_x u$
- NC Boussinesq. $u_{tt} = u_{xxx} + (u^2)_x - [u, \partial_x^{-1} u_t]_x$
- NC NLS eq. $i\psi_t = \psi_{xx} - \varepsilon \psi \bar{\psi} \psi$
- NC mKdV eq. $v_t = v_{xxx} - v^2 v_x - v_x v^2$
- NC Sawada-Kotera eq. summarized in [H, hep-th/0601209
H, hep-th/0507112]

$$u_t + u_{5x} + u_{xxx} u + u_{xx} u_x + u u_x u = 0$$

- NC Toda, sine-Gordon, Tzitzéica, ...

§3 ADHM Construction of Instantons

Instantons are finite-action solutions of ASDYM



codim 4 solitons in 4-dim = instantons

ADHM construction is a powerful method

$\begin{cases} \text{to give any instantons} \\ \text{based on 1:1 correspondence between} \end{cases}$

instanton moduli sp. & ADHM moduli sp.

$G = U(N)$ k-instanton

A_μ: $N\times N$ $\quad \underset{\uparrow}{k}$ $\frac{1}{8\pi^2}\int F\wedge F = k$

sol. of ASDYM eq.

(mod $U(N)$ gauge trf.)

$\longleftrightarrow$

1:1

"$G = U(k)$" ADHM data

$B_{1,2}$, I, J

$k\times k$ $\quad k\times N$ $\quad N\times k$

sol. of ADHM eq.

(mod "$U(k)$ gauge trf")

$$*F_{\mu\nu} = -F_{\mu\nu}$$

(PDE in 4dim)

$$\begin{cases} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + I I^\dagger - J^\dagger J = 0 \\ [B_1, B_2] + IJ = 0 \end{cases}$$

(Matrix eq.)

Difficult

easy

[Atiyah-Drinfeld-Hitchin-Manin, PLA65('78)185]

$G = U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\leftrightarrow$

1:1

"$G = U(1)$" ADHM data

$B_{1,2}$, I , J

$k\times k$ $k\times N$ $N\times k$

$\shortparallel$ $\shortparallel$ $\shortparallel$

1×1 1×2 2×1

$$\begin{cases} \underbrace{[B_1, B_1^\dagger] + [B_2, B_2^\dagger]} + II^\dagger - J^\dagger J = 0 & k\times k \\ \underbrace{[B_1, B_2]}_{\text{dropped}} + IJ = 0 & k\times k \end{cases}$$

$k = 1$

Difficult

easy

$G = U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$ 1:1

"$G = U(1)$" ADHM data

$B_{1,2}$ = arbitrary

$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} I = \begin{bmatrix} \rho & 0 \end{bmatrix} \\ J = \begin{bmatrix} 0 \\ \rho \end{bmatrix} \end{cases} \qquad \rho \in \mathbb{R}$$

Difficult

easy

$G=U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\overset{}{\longleftrightarrow}$ 1:1

"$G=U(1)$" ADHM data

$$B_{1,2} = \alpha_{1,2} \in \mathbb{C}$$

$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$

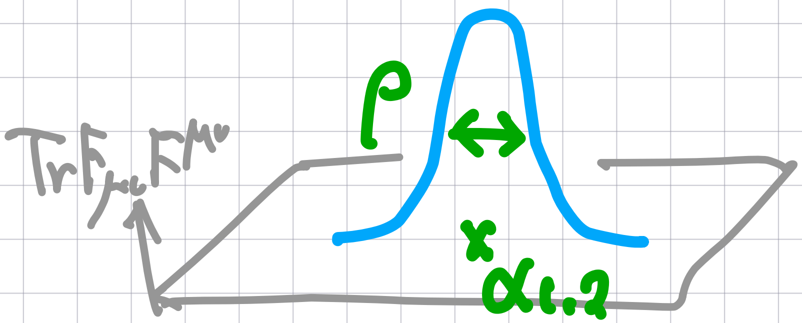
$$\Downarrow$$

$$\begin{cases} I = \boxed{\rho \mid 0} \\ J = \boxed{\begin{matrix} 0 \\ \rho \end{matrix}} \end{cases} \qquad \rho \in \mathbb{R}$$

$$\longleftarrow$$

$$A_\mu = \frac{i(x-b)^\nu \eta^{(-)a}_{\mu\nu} \sigma^a}{|z-\alpha|^2 + \rho^2}$$

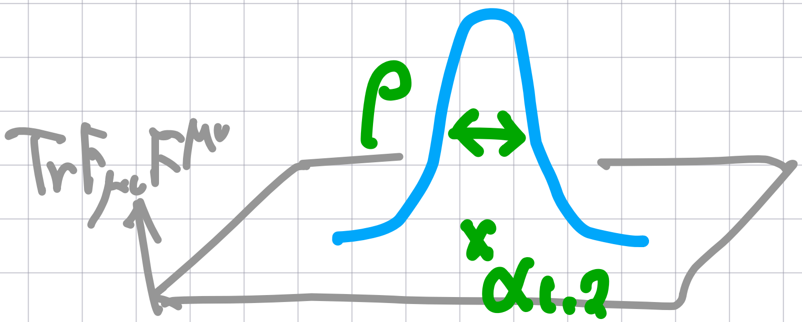
BPST instanton!



$G=U(2)$ 1-instanton

"$G=U(1)$" ADHM data

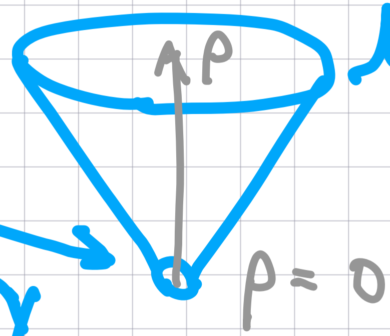
$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$



$B_{1,2} = \alpha_{1,2} \in \mathbb{C}$

$I = \begin{pmatrix} \rho & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$

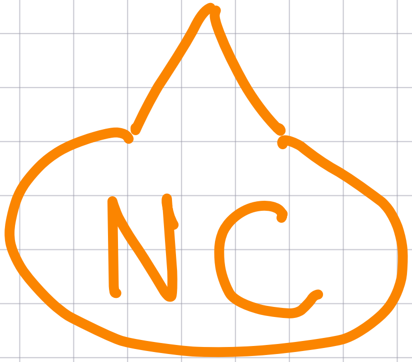
· Let's take $\rho \to 0$ limit:



$\mathcal{M}$ (moduli sp)

small instanton singularity

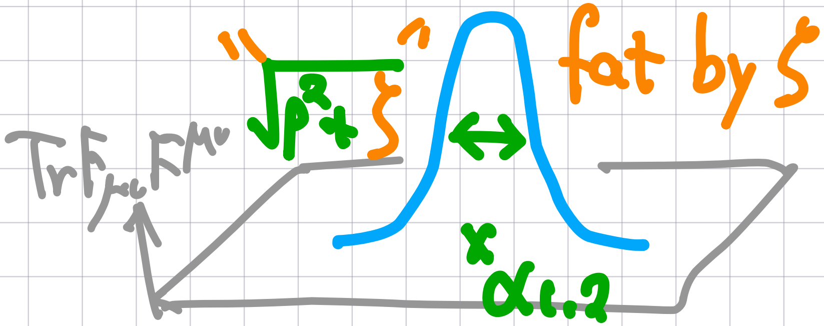
$G=U(2)$ 1-instanton



"$G=U(1)$" ADHM data

$[B_1, B_1^\dagger] + \underbrace{[z_1, \bar{z}_1]}_{} + \cdots$

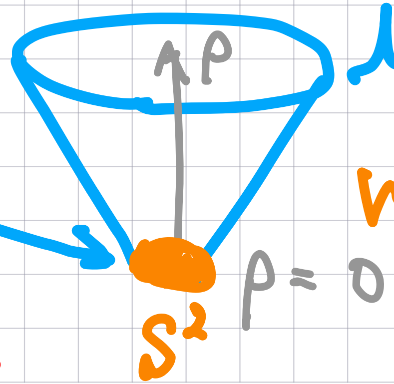
$$\begin{cases} II^\dagger - J^\dagger J = \zeta \ (>0) \\ IJ = 0 \end{cases}$$



$B_{1,2} = \alpha_{1,2} \in \mathbb{C}$

$I = \begin{pmatrix} \sqrt{\rho^2+\zeta} & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$

- Even in $\rho \to 0$ limit, configuration is smooth (U(1) instanton) special to NC space!



$\mathcal{M}$ (moduli sp)

resolution of singularity

D-brane interpretation = kD0-ND4 BPS branes

$G=U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$ 1:1 (trivial!?)

"$G=U(1)$" ADHM data

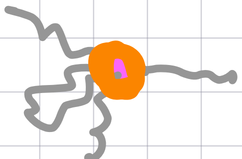
$B_{1,2}$ = arbitrary

$$\begin{cases} II^\dagger - J^\dagger J = 0 \\ IJ = 0 \end{cases}$$

BPS condition
=
ASDYM

2 = N D4-branes

1 = k D0-branes



open strings

BPS condition
=
ADHM eq

[Douglas, hep-th/9512077]
[Witten, hep-th/9511030]

D-brane interpretation = k D0-N D4 BPS branes

[Nekrasov-Schwarz CMP 198 ('98) 689]

$G = U(2)$ 1-instanton

$$*F_{\mu\nu} = -F_{\mu\nu}$$

$\longleftrightarrow$ 1:1 (trivial!?)

"$G = U(1)$" ADHM data

$B_{1,2}$ = arbitrary

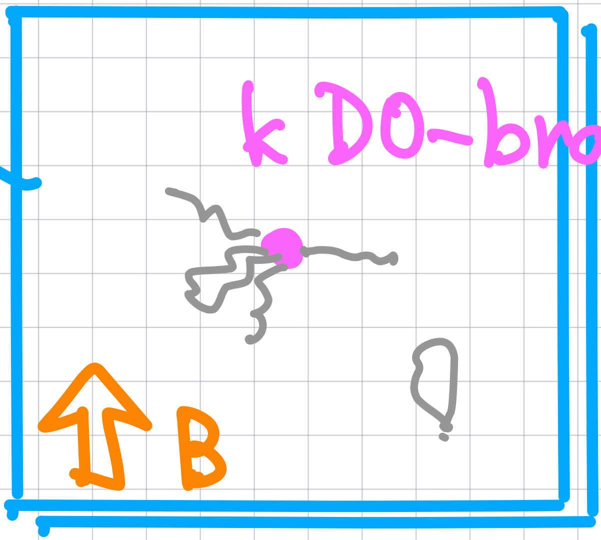
$$\begin{cases} II^{\dagger} - J^{\dagger}J = \zeta \\ IJ = 0 \end{cases}$$

ζ: FI param.

BPS condition = ASDYM

BPS condition = ADHM eq

turn on bkg B-field



N D4-branes

§4 NC KdV eqs. & Quasideterminants

NC Korteweg-de Vries (KdV) eq.

$$4u_t + u_{xxx} + 3(u \star u_x + u_x \star u) = 0$$

$[t, x] = i\theta$

Moyal products

$$\underset{\text{c-number}}{\underbrace{f(x)}} \star \underbrace{g(x)} := f(x)\, e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}\, g(x)$$

c-number

$$= f(x)\cdot g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \cdot \partial_\nu g + O(\theta^2)$$

$$[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = x^\mu \cdot x^\nu + \frac{i}{2}\theta^{\mu\nu} - \left(x^\nu \cdot x^\mu + \frac{i}{2}\theta^{\nu\mu}\right) = i\theta^{\mu\nu}$$

NC KdV eq. has infinite conserved densities :

$$4u_t + u_{xxx} + 3(u u_x + u_x u) = 0$$ (products are all Moyal)

$[t, x] = i\theta$

$$\sigma_n = \mathrm{coef}_{-1} \tilde{L}^n - 3\theta\left((\mathrm{coef}_{-1}\tilde{L}^n) \diamond u_3' + (\mathrm{coef}_{-2}\tilde{L}^n) \diamond u_2' \right)$$

Strachan's product (comm. nonassoc.)

$$\text{w/ } \tilde{L} = \sqrt{\partial_x^2 + u} = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + \cdots$$

$$\mathrm{coef}_{-k} \tilde{L}^n = \left(\cdots + \underset{\text{this one}}{\boxed{}}\, \partial^{-k} + \cdots = \tilde{L}^n\right)$$

[Dimakis & Müller-Hoissen, hep-th/0007074]

[H. hep-th/0311206] NC KP/GD hierarchies

- suggests infinite symmetry
- Significance is not clarified

Lax representation

spectral parameter

$$\begin{cases} Lf = 0 \\ Mf = 0 \end{cases} \cdots ① \quad w/ \quad \begin{cases} L := \partial_x^2 + u - \zeta \\ M := \partial_t + 4\partial_x^3 + 6u\,\partial_x + 3u_x \end{cases}$$

- Compatible condition of ① $[L, M] = 0 \Leftrightarrow$ NC KdV eq.

Darboux transformation

$\zeta = \lambda$ (fix)

Prepare a special sol. of ① : $\begin{cases} L\theta = 0 \\ M\theta = 0 \end{cases}$ $\quad \theta = f(\lambda)$

The following trf. leaves ① as it is:

$$(D) \quad \begin{cases} L \mapsto \tilde{L} = G_\theta L G_\theta^{-1} \\ M \mapsto \tilde{M} = G_\theta M G_\theta^{-1} \\ f \mapsto \tilde{f} = G_\theta f \end{cases} \qquad \begin{aligned} G_\theta &:= \theta\,\partial_x\,\theta^{-1} \cdots ② \\ &= \partial_x - \theta_x\,\theta^{-1} \end{aligned}$$

The Darboux trf (D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$

n-iterations of (D) from a trivial seed sol.

$$\begin{cases} (\partial_x^2 - \lambda_k)\,\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\,\theta_k = 0 \end{cases} \qquad k \in \{1, \cdots, n\}$$

$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

derivative

[Etingof-Gelfand-Retakh, q-alg/9701008]
[Gilson-Nimmo, nlin.SI/0701027]

Quasideterminant

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := d - CA^{-1}B \xrightarrow[d:1\times 1]{\text{commutative limit}} \frac{\begin{vmatrix} A & B \\ C & d \end{vmatrix}}{|A|}$$

squares

(Schur complement)

The Darboux trf (D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$

n-iterations of (D) from a trivial seed sol.

$$\begin{cases} (\partial_x^2 - \lambda_k)\theta_k = 0 \\ (\partial_t + 4\partial_x^3)\theta_k = 0 \end{cases} \quad k \in \{1, \cdots, n\}$$

$$u_n = -2\partial_x \begin{vmatrix} \theta_1 & \cdots & \theta_n & 0 \\ \theta_1' & \cdots & \theta_n' & 0 \\ \vdots & & \vdots & \vdots \\ & & & 0 \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 1 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\xrightarrow[\text{limit}]{\text{com.}} 2\partial_x \frac{\partial_x Wr(\theta_1, \cdots, \theta_n)}{Wr(\theta_1, \cdots, \theta_n)} = 2\partial_x^2 \log \overbrace{Wr(\theta_1, \cdots, \theta_n)}^{\tau\text{-fcn}}$$

det $\square$

Hirota trf.

Quasideterminant

- Not just a NC generalization
- But rather relates to inverse matrix

Def [Gelfand–Retakh, 1991]

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}, \quad Y = X^{-1} = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix}$$

$$\Rightarrow \quad |X|_{ij} := y_{ji}^{-1} \qquad \left(\underset{\text{limit}}{\overset{\text{com.}}{=}} \frac{1}{\det X^{ij}} \cdot \det X \right)$$

(i,j)-th quasideterminant

Examples (recursively obtained)

$n=1$) $\quad |X|_{11} = x_{11}$

Recall : $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow X^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DBA)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$

[**New symbol** : $|X|_{ij} \equiv \begin{vmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{vmatrix}$]

$n=2$) $\quad \begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12}\, x_{22}^{-1}\, x_{21}$,

$\begin{vmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11}\, x_{21}^{-1}\, x_{22}$, $\quad \cdots$

[New symbol : $|X|_{ij} \equiv \left| \begin{matrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{matrix} \right|$]

n = 1) $\quad |X|_{11} = x_{11}$

Recall : $X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \Rightarrow X^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DBA)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$

n = 2) $\quad \begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12}\, x_{22}^{-1}\, x_{21} ,$

$\begin{vmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11}\, x_{21}^{-1}\, x_{22} , \quad \cdots$

n = 3) $\quad \begin{array}{ccc} A & & B \\ & \left| \begin{array}{c|cc} \boxed{x_{11}} & x_{12} & x_{13} \\ \hline x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{array} \right| & \\ C & & D \end{array} = x_{11} - (x_{12}\ x_{13}) \underbrace{\begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}^{-1}}_{\text{can be written by 2x2 results}} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix}$

Some formula

Prop

$k=1,\cdots,n-1$

$$\begin{vmatrix} a_{11} & \cdots & a_{k1}\lambda & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \cdots & a_{kn}\lambda & \cdots & \boxed{a_{nn}} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{k1} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \cdots & a_{kn} & \cdots & \boxed{a_{nn}} \end{vmatrix}$$

NC Jacobi id.

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}$$

Very powerful!

Homological rel.

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}$$

Soliton Scattering

[H. hep-th/0610006.]
[H. arXiv: 1806.05188]

Bad News when $f(x)$ and $g(x)$ are real, $f(x) \star g(x)$ is not in general

The solutions could be real in some cases:

(i) 1-soliton solutions are the same as commutative ones because of

- $$f(x-vt) \star g(x-vt) \equiv f(x-vt) \cdot g(x-vt) \quad \forall v$$

$[t, x] = i\theta$

(ii) N-soliton solutions in the asymptotic region

Ride on a comoving frame with I-th soliton:

$$\tilde{x} := x - 4\lambda_I t \ : \text{ finite}$$

$$x - 4\lambda_k t = \underbrace{x - 4\lambda_I t}_{\tilde{x}} + 4(\lambda_I - \lambda_k)t$$

$$4(\lambda_I - \lambda_k)t \xrightarrow{t \to \pm\infty} \pm\infty$$

Assume $0 < \lambda_1 < \cdots < \lambda_I < \cdots < \lambda_n$

$$\theta_k \xrightarrow[t\to\infty]{} \begin{cases} e^{-\sqrt{\lambda_k}(x-4\lambda_k)t} & k < I \\ e^{+\sqrt{\lambda_k}(x-4\lambda_k)t} + e^{-\sqrt{\lambda_k}(x-4\lambda_k)t} & k = I \\ e^{+\sqrt{\lambda_k}(x-4\lambda_k)t} & k > I \end{cases}$$

← I forgot "4"...

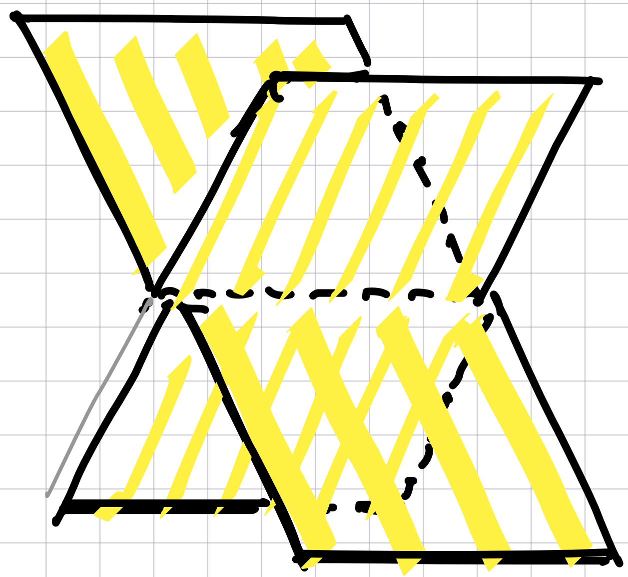
$$
u \xrightarrow{t\to\infty}
\begin{vmatrix}
1 & \cdots & 1 & e^{\sqrt{\lambda_I}(x-\lambda_I t)} + e^{-\sqrt{\lambda_I}(x-\lambda_I t)} & \cdots & 1 & 0 \\
(-\lambda_1^{\frac{1}{2}}) & & (-\lambda_{I-1}^{\frac{1}{2}}) & \sqrt{\lambda_I}\left(e^{\sqrt{\lambda_I}(x-\lambda_I t)} - e^{-\sqrt{\lambda_I}(x-\lambda_I t)}\right) & & (\lambda_n^{\frac{1}{2}}) & 0 \\
(-\lambda_1^{\frac{1}{2}})^2 & & (-\lambda_{I-1}^{\frac{1}{2}})^2 & \vdots & & (\lambda_n^{\frac{1}{2}})^2 & \vdots \\
\vdots & & \vdots & \vdots & & \vdots & 0 \\
 & & & & & & 1 \\
(-\lambda_1^{\frac{1}{2}})^n & \cdots & (-\lambda_{I-1}^{\frac{1}{2}})^n & \underbrace{\sqrt{\lambda_I}^{\,n}\left(\qquad\qquad \right)}_{x\text{-dependence is here only "}f(x-\lambda_I t)\text{"}} & & (\lambda_n^{\frac{1}{2}})^n & \boxed{0}
\end{vmatrix}
$$

⇒ In $t \to \pm\infty$ (asymptotic region), Moyal products reduce to ordinary products (coincides with commutative ones)

NC n-soliton solution (asymptotically)

= "non-linear superposition" of n 1-solitons

(with the same phase shifts as commutative case)



§5 ASDYM revisited

We present soliton solutions of ASDYM whose codim is one (not 4 : instantons)

TMP 122, 284
[Nimmo, Gilson, Ohta, '00]

These are described in terms of

quasi-Wronskian generated by Darboux trf.
or
quasi-Grammian " binary Darboux trf.

In KP theory, τ-functions are Wronskians (= Grammians)

Here we show the Grammian solutions.

ASDYM eq. $F_{zw}=0,\ F_{\tilde{z}\tilde{w}}=0,\ F_{z\tilde{z}}-F_{w\tilde{w}}=0$

has two potential forms:

- J-matrix form (Yang's eq.)

$$\partial_{\tilde{z}}\left((\partial_z J)J^{-1}\right)-\partial_{\tilde{w}}\left((\partial_w J)J^{-1}\right)=0 \quad \cdots ①$$

- K-matrix form (Newman's eq.)

$$\partial_z\partial_{\tilde{z}}K-\partial_w\partial_{\tilde{w}}K+[\partial_w K,\partial_z K]=0 \quad \cdots ②$$

- There exist actions whose EoM are ① or ②
- Choose: $z=x^1+x^3,\ w=x^2+x^4,\ \cdots\in\mathbb{R} \Rightarrow (++--)$

4-dim WZW model (J-matrix)

[Donaldson '85]

Action: $S_{4dWZW} = S_\sigma + S_{WZ}$ $\quad J(x) \in GL(N,\mathbb{C})$

$$S_\sigma = \underline{\frac{i}{4\pi}} \int_{\mathbb{U}} \underbrace{\omega \wedge \mathrm{Tr}\left[(\partial J)J^{-1} \wedge (\tilde{\partial} J)J^{-1}\right]}_{\mathcal{L}_\sigma}$$

$$S_{WZ} = -\underline{\frac{i}{12\pi}} \int_{\mathbb{U}} \underbrace{A \wedge \mathrm{Tr}\left[(dJ)J^{-1}\right]^3}_{\mathcal{L}_{WZ}}$$

$(z, w, \tilde{z}, \tilde{w})$: local coords of $\mathbb{U}$

w/ $\omega = dA$: Kähler form of M_4

$\mathbb{U}$: $z = x^1 + x^3,\ w = x^2 + x^4, \ldots$ $\qquad \omega = \frac{i}{2}(dz \wedge d\tilde{z} - dw \wedge d\tilde{w})$

$$d = \partial + \tilde{\partial}, \quad \partial = dw\,\partial_w + dz\,\partial_z, \quad \tilde{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$$

EoM: $\partial_{\tilde{z}}\left((\partial_z J)J^{-1}\right) - \partial_{\tilde{w}}\left((\partial_w J)J^{-1}\right) = 0$ Yang eq. (= ASDYM)

n-soliton sols. for $G = U(2)$:

PRD 111, 086023 [Li-H.-Huang-Zhang '25]

$$\underset{2\times 2}{J} = \begin{vmatrix} \Omega(\theta,\theta) & \Lambda^{-1}\theta^{\dagger} \\ \theta & \boxed{1_2} \end{vmatrix}$$

← quasi-Grammian type (One binary Darboux trf.)

$$\underset{2\times n}{\theta} = \begin{pmatrix} a_1^2 e^{\xi_1} & a_2^2 e^{\xi_2} & \cdots & a_n^2 e^{\xi_n} \\ b_1^2 e^{-\xi_1} & b_2^2 e^{-\xi_2} & \cdots & b_n^2 e^{-\xi_n} \end{pmatrix}, \quad \underset{n\times n}{\Lambda} = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

$$\xi_k = \underbrace{\lambda_k \alpha_k z + \beta_k \tilde{z}}_{} + \underbrace{\lambda_k \beta_k w + \alpha_k \tilde{w}}_{}$$

linear in spacial coords

sols. of $\theta_z = \theta_{\tilde{w}} \Lambda$, $\theta_w = \theta_{\tilde{z}} \Lambda$

(Ω: sol. of $\Lambda^{\dagger}\Omega - \Omega\Lambda = \theta^{\dagger}\theta$)

※ Quasideterminant (Schur complement)

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := d - CA^{-1}B \qquad (d, A: \text{square matrices})$$

Action Density $\mathcal{L}_{\text{4dWZW}} = \mathcal{L}_a + \mathcal{L}_{\text{WZ}}$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

One soliton (on $\mathbb{U}$)

$$\mathcal{L}_a = \frac{1}{8\pi} d_{11} \operatorname{sech}^2 X$$

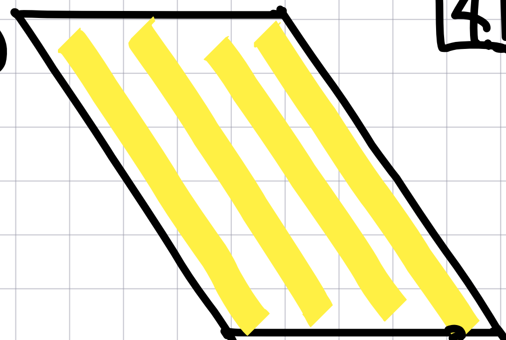
$X := \xi + \bar{\xi}$: linear in x^μ & real

$$\mathcal{L}_{\text{WZ}} \equiv 0$$ (identically)

peak

$X = 0$

4D



3-dim hyperplane (codim 1)

Similar!

not instanton!

cf. KP soliton

$$u = 2\partial_x^2 \log\left(e^X + e^{-X}\right) \propto \operatorname{sech}^2 X$$

linear in t, x, y

Two Soliton (on III)

$X_k = \xi_k + \bar{\xi}_k$, $\Theta_{12} = \Theta_1 - \Theta_2$

$i\Theta_k = \xi_k - \bar{\xi}_k$

$$\mathcal{L}_a = \frac{\left[\begin{array}{l} A\cosh^2 X_1 + B\cosh^2 X_2 \\ + C_\pm \cosh^2\left(\frac{X_1+X_2 \pm i\Theta_{12}}{2}\right) + D_\pm \cosh^2\left(\frac{X_1 - X_2 \pm i\Theta_{12}}{2}\right)\end{array}\right]}{2\pi\left(a\cosh(X_1+X_2) + b\cosh(X_1 - X_2) + c\cos\Theta_{12}\right)^2}$$

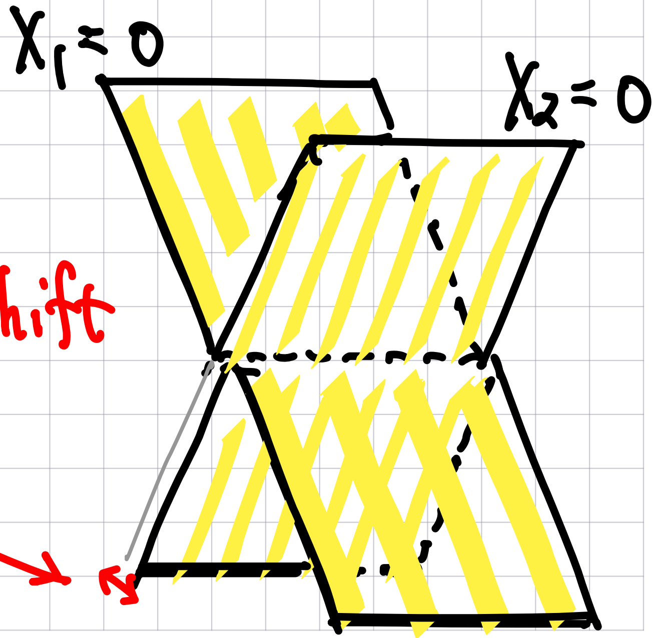
$$\xrightarrow[X_1:\text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_1 \pm \delta)$$

non-singular

$$\xrightarrow[X_2:\text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_2 \pm \delta)$$

phase shift

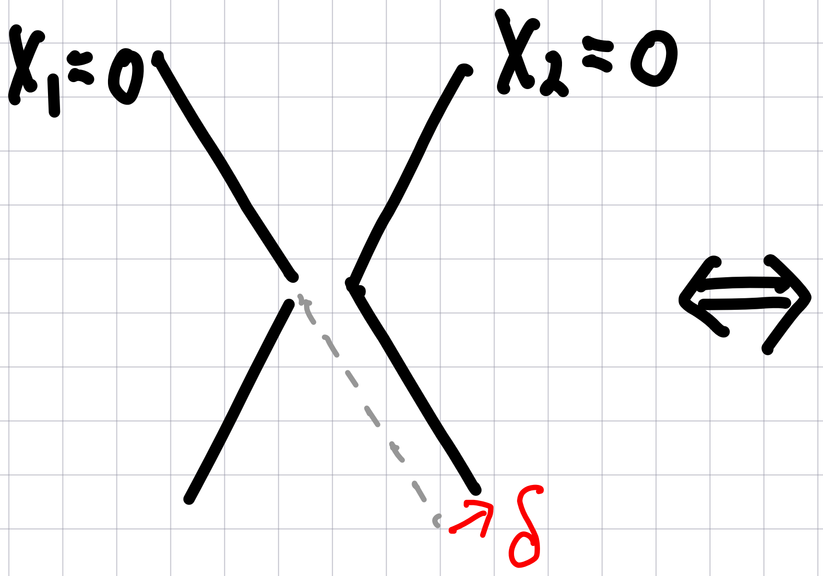
$$\xrightarrow[\text{otherwise}]{r\to\infty} 0$$

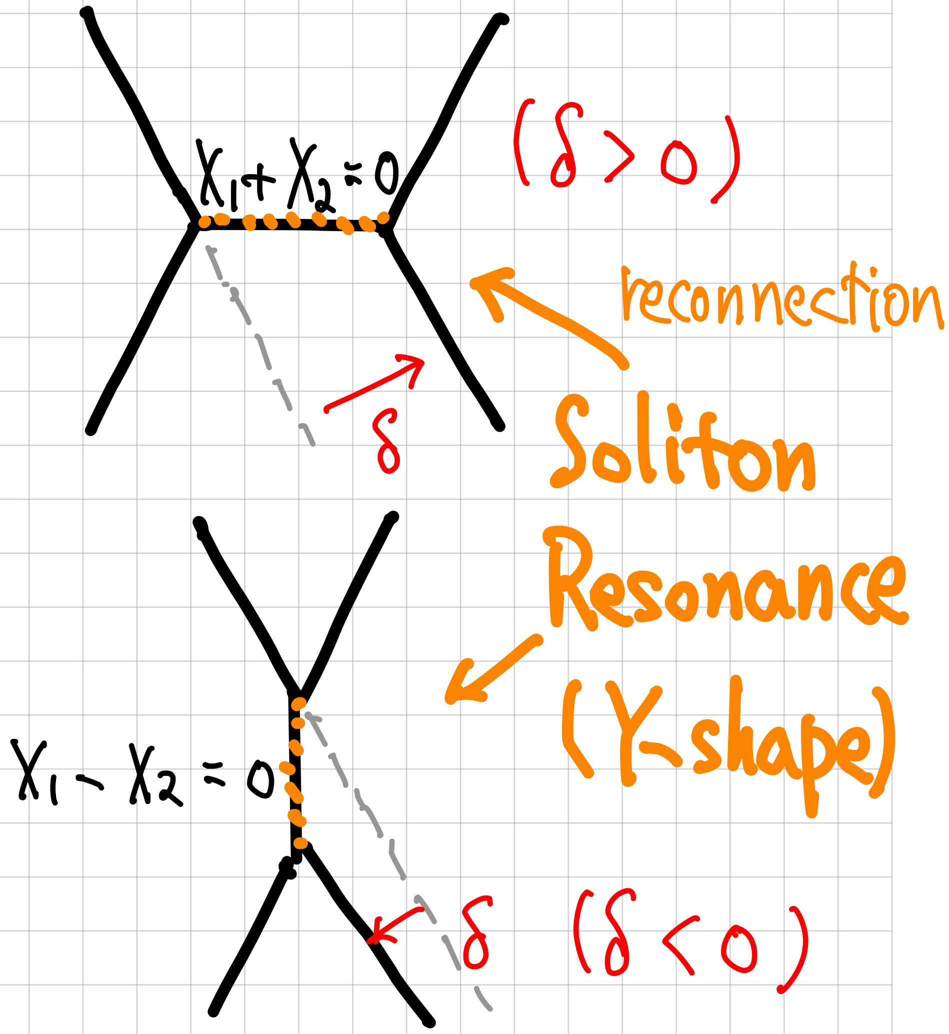


$\mathcal{L}_{WZ} = \begin{pmatrix} \text{very} \\ \text{long} \end{pmatrix}$

(non-linear effect)

※ Origin of Phase shift (KP case)





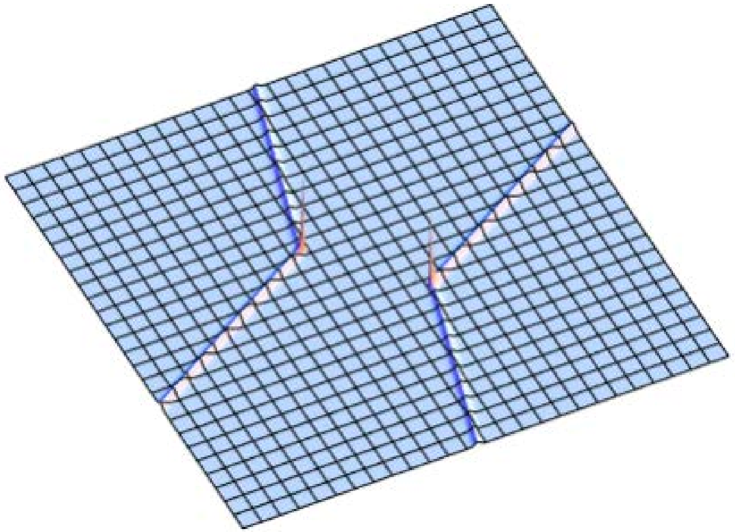
"Intermediate States" appear in large δ

(Seen at shallow water waves)

[Book of Yuji Kodama]

(i) large δ (>0)

$$\delta = \log \frac{|\lambda_1 - \lambda_2|}{|\lambda_1 - \bar{\lambda}_2|}$$



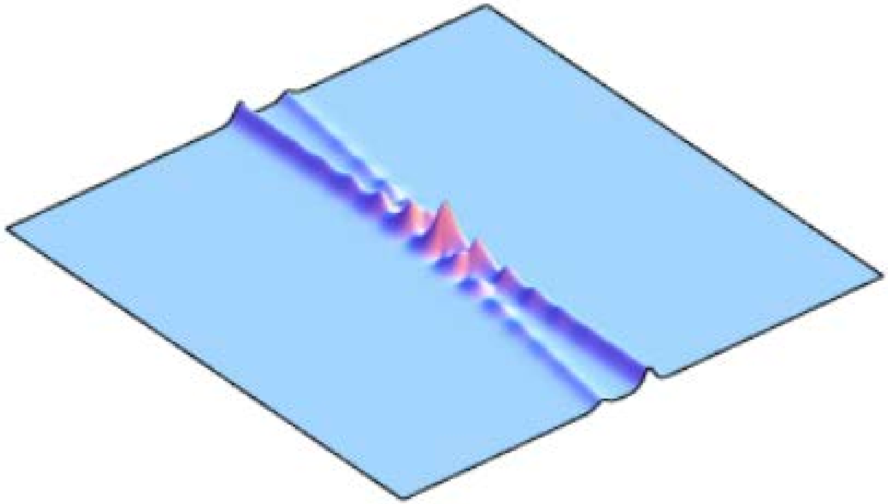
Not 2 Y-shaped soliton

But 2 V-shaped soliton! (new)

(Exists in 2+1 dim CBS eq) [Yu-Toda-Sasa-Fukuyama '98]

(ii) large δ (<0)

with $\begin{matrix} \alpha_1 = \alpha_2 \\ \beta_1 = \beta_2 \end{matrix} \left(\begin{matrix} \Leftrightarrow \\ \xi_1 \to \xi_2 \end{matrix} \right)$



Double pole solution (resonance)

※ Can be constructed directly from

$$\Theta = \begin{pmatrix} \xi e^{\xi} & e^{\xi} \\ -\xi e^{-\xi} & e^{-\xi} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$$

Jordan form

Summary

- Classification of KP soliton [Yuji Kodama et al.]

  Soliton graphs based on Y-shaped parts

  $\leftrightarrow$ positive Grassmannian

- Classification of ASDYM soliton

  Soliton planes based on V-shaped parts

  $\leftrightarrow$?? $\rightsquigarrow$ moduli, $N=2$ string, $\cdots$

cf. S. Kakei, Ann. Henri Poincaré 3 ('02) 817

Towards Unification of Integrable Systems

4d ASDYM $\underset{(++--)}{\text{田}}$

$\downarrow$

various integrable eqs.

(KdV NLS, Toda, ⋯)

[Ward, Mason–Woodhouse]

⋯

Towards Unification of Integrable Systems

4d Chern-Simons (CS)

↓

various

solvable models

(spin chains, PCM, ⋯)

[Costello-Yamazaki-Witten, ⋯]

(2017 & 18)

4d ASDYM $\overline{U}$ $(++--)$

↓

various

integrable eqs.

(KdV NLS, Toda, ⋯)

$\underline{\text{Ex}}$ 4dCS → Principal Chiral Model (PCM) in 2d

[Costello-Yamazaki '19]

$$S_{4dCS} = \bullet \int_{\mathbb{R}^2 \times \mathbb{CP}^1} \omega_1 \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

[Delduc-Lacroix-Magro-Vicedo, 1909.13824]

$\mathbb{R}^2 \ni (\sigma^+, \sigma^-)$, $\mathbb{CP}^1 \ni (z, \bar{z})$ ← spectral parameter

$$\omega_1 = \underbrace{\varphi(z)}_{\text{has poles}} dz \qquad \text{i.e. } \varphi(z) = \frac{1-z^2}{z^2} dz$$

← double poles at $z = 0, \infty$

simple zero at $z = \pm 1$

$$A = A_+ d\sigma^+ + A_- d\sigma^- + A_{\bar{z}} d\bar{z}$$ ← $A_z = 0$ (gauge fix)

(bulk) EoM: $\omega \wedge F = 0$ w/ b.c. at the poles

$$A = -d\hat{g}\,\hat{g}^{-1} + \hat{g}\,\mathcal{L}\,\hat{g}^{-1}, \quad \partial_{\bar{z}} \mathcal{L} = 0$$

$\hat{g}(z=0) = g$

$\hat{g}(z=\infty) = 0$

$+ \; -$

$$\mathcal{L} = \underbrace{-\frac{1}{z+1} g^{-1}\partial_+ g \, d\sigma^+}_{\mathcal{L}_+} + \underbrace{\frac{1}{z-1} g^{-1}\partial_- g \, d\sigma^-}_{\mathcal{L}_-}$$

Lax pair of PCM!

Lax formalism of ASDYM eq.

linear system

spectral parameter $\in \mathbb{C}P_1$

$$\begin{cases} Lf = (D_w - \zeta D_{\tilde{z}}) f = 0 \\ Mf = (D_z - \zeta D_{\tilde{w}}) f = 0 \end{cases}$$

$D_w = \partial_w + A_w, \cdots$

$f = f(x;\zeta)$

Compatible condition $[L, M] = 0 \Leftrightarrow$ ASDYM

☺ We note that $F_{\mu\nu} = [D_\mu, D_\nu]$

$$[L, M] = \underbrace{[D_w, D_z]}_{F_{wz}} - (\underbrace{[D_{\tilde{z}}, D_z] + [D_w, D_{\tilde{w}}]}_{-F_{z\tilde{z}} + F_{w\tilde{w}}})\zeta + \underbrace{[D_{\tilde{z}}, D_{\tilde{w}}]}_{F_{\tilde{z}\tilde{w}}}\zeta^2 = 0 \quad ({}^\forall \zeta)$$

$\blacksquare$

Towards Unification of Integrable Systems

4d Chern-Simons (CS)

$\downarrow$

various

solvable models

(spin chains, PCM, ⋯)

[Costello-Yamazaki-Witten, ⋯]
(2017)

4d ASDYM $\underset{(++--)}{\text{ひ}}$

$\downarrow$

various

integrable eqs.

(KdV NLS, Toda, ⋯)

Towards Unification of Integrable Systems

6d CS

[Bittleston–Skinner, 2023]
JHEP 02, 227

↙ ↘

4d CS ← duality relation → 4d ASDYM $\mathbb{U}$ $(++--)$

↓ ↓

various solvable models (spin chains, PCM, ...)

various integrable eqs. (KdV, NLS, Toda, ...)

Towards Unification of Integrable Systems

6d CS

↙ ↘

4d CS

↓

various solvable models

(spin chains, PCM, …)

4d ASDYM hier. on $\mathbb{U}$ (++−−)

↓

various integrable eqs.

(KdV NLS, Toda, …)

KP hier.

[Li-Maruno-Zhang, arXiv:2603.23060]

my postdoc. →

Towards Unification of Integrable Systems

6d CS

NC 5d CS
(Bittleston-Heuveline-Raghevendran-Skinner] [arXiv:2509.20643]

4d CS

various solvable models
(spin chains, PCM, ⋯)

KP eq !

4d ASDYM
(+ + − −)
hier.

various integrable eqs.
(KdV NLS, Toda, ⋯)
KP hier.
[Li-Maruno-Zhang, arXiv:2603.23060]

my postdoc. → ~

Towards Unification of Integrable Systems

6d CS

com. 5d CS

4d ASDYM U (++--)

4d CS

dispersionless KP eq

hier.

various solvable models (spin chains, PCM, …)

various integrable eqs. (KdV NLS, Toda, …) KP hier.

[Li-Maruno-Zhang, arXiv:2603.23060]

my postdoc. → ~

Towards Unification of Integrable Systems

NC 6d CS

NC 5d CS

NC 4d CS

various NC solvable models

NC (spin chains, PCM, ...)

KP eq !

NC

4d ASDYM

U (++−−)

hier.

various NC integrable eqs.

NC (KdV, NLS, Toda, ...)

NC KP hier.

God's blessing (after 2023.3)

- Many suggestions on higher-dim extension of KP theory or Sato/Kodama theory
- Many students & postdocs join my group (8 graduate students & 2 postdocs)
- 10 Grants approved (crazy)

Note In Japan, everything can be god after death

God (= Mikio Sato) blessing (after 2023.3)

passed away 2023.1 consistent!

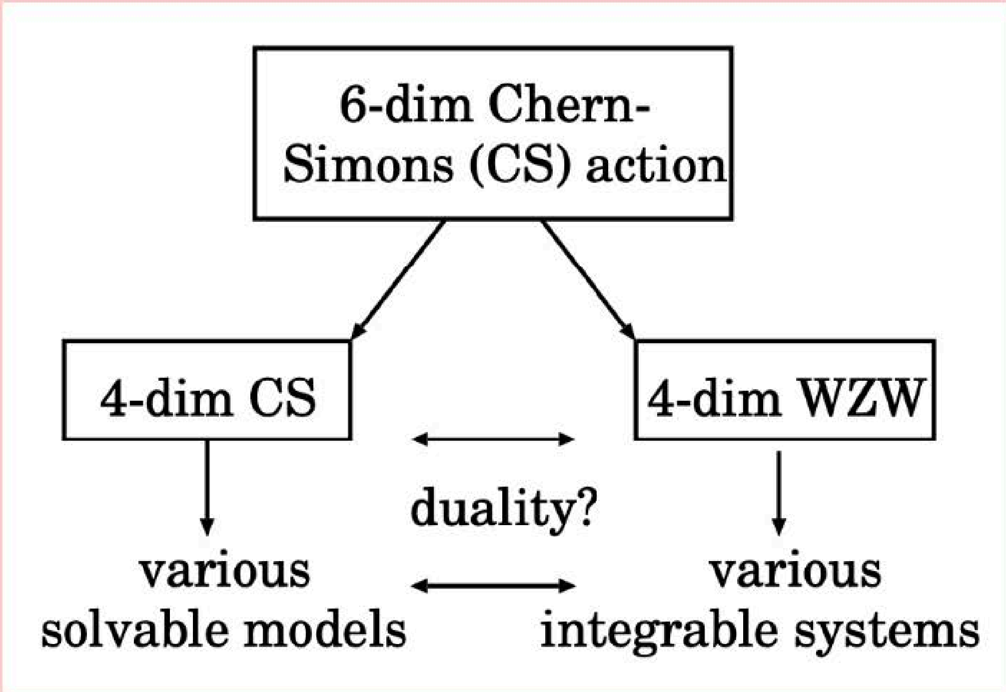
- Many suggestions on higher-dim extension of KP theory or Sato / Kodama theory
- Many students & postdocs join my group (8 graduate students & 2 postdocs)
- 10 Grants approved (crazy)

Note In Japan, everything can be god after death

Workshop on the Unification (2025.10 Nagoya)

Unification of Integrable Systems

27〜29 October, 2025 at Nagoya University and Online



Workshop on the Unification 2 (2026. Fall Osaka)

Daiwa Angro-Japanese Foundation

Unification of Integrable Systems 2

Autumn, 2026 in Osaka and Online

Workshop on the Unification 3 (2027 Summer York, UK)

Daiwa Angro-Japanese Foundation

Unification of Integrable Systems 3

Summer 2027 in York and Online?

Welcome to Join!

Thank You

Very Much

References (Reviews)

- NC integrable system : [H, arXiv:1012.6043]
- NC instantons : [H-Nakatsu, arXiv:1311.5227]
- Quasideterminants : [GGRW, math/0208146]

Useful Slides & Videos

※ Workshop on NC integrable systems

 & on Unification of Integrable Systems

 Nagoya Math-Phys Seminars

Workshop on homotopy alg. ... (2026.8 Nagoya)

(on occasion of Ivo Sachs's visit)

Nagoya University International Conference on

Homotopy Algebraic Formulation of SFT, QFT and IFT

24～26 August, 2026 at Nagoya University and Online

Supported by Nagoya University International Conference Grant and Mugishima Foundation.

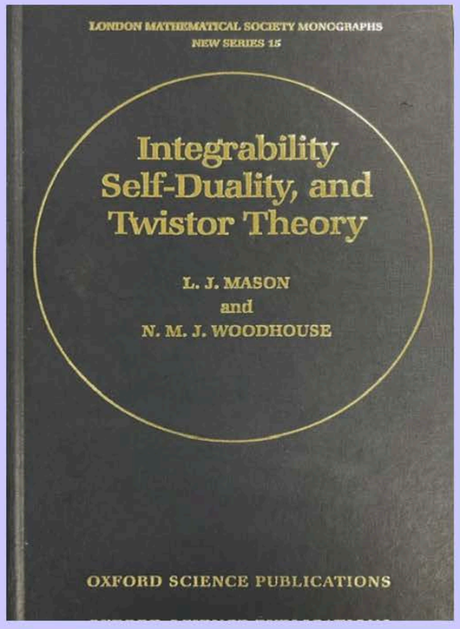
Workshop on twistor

(2026.8 Nagoya)

(on occassion of Lionel Mason's visit)

Integrability, Self-Duality, and Twistor Theory

27〜28 August, 2026 at Nagoya University and Online



This workshop is co-sponsored by [Daiko Foundation] and ...