Hilbert transform on lattices of $\mathrm{PSL}_2(\mathbf{C})$

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Fourier multiplier: $T_m : \int_{\mathbf{R}^n} \widehat{f}(y)\, e^{\cdot y i}\, dy \mapsto \int_{\mathbf{R}^n} m(y) \widehat{f}(y)\, e^{\cdot y i}\, dy$

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$\partial\Omega$ is piecewise flat

m is a combination of Hilbert transforms

$\Downarrow$

T_m is bounded

$\partial\Omega$ is curved somewhere

Fefferman's ball theorem

$\Downarrow$

T_m is unbounded

1. The simplest way to prove Hilbert transform's L_p-boundedness is the following

Theorem: Cotlar's identity (1955)

If m is a bounded, real-valued symbol satisfying

$$\big(m(x) - m(-y)\big)\big(m(x+y) - m(y)\big) = 0, \quad x, y \in \mathbf{R}^n$$

then T_m is a bounded L_p-multiplier, $1 < p < \infty$.

2. The unbounded case is much more involved (Kakeya constructions...)

Fourier analysis means decomposing functions into waves, like

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i x n}$$

For G a group, we have natural notion of NC waves $\lambda_g \in B(L_2 G)$ of frequency $g \in G$,

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Remark: you also have a notion of integral $f \mapsto \widehat{f}(e)$, so you can talk of $L_p(\widehat{G})$.

Definition: a Fourier multiplier is an operator that acts on frequencies like

$$T_m(f) = \sum_{g \in G} m(g) \widehat{f}(g) \lambda_g$$

Goal: find conditions on m to decide if T_m is L_p-bounded or unbounded

Ball theorem for Lie groups [Parcet, de la Salle, Tablate 2024]

T_m idempotent multiplier, locally L_p-bounded for some $p \neq 2$

$$\Downarrow$$

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Cotlar identity for Groups

If m is bounded, real-valued and

$$\big(m(g^{-1}) - m(h)\big)\big(m(gh) - m(g)\big) = 0, \quad g, h \in G$$

then T_m is L_p-bounded for $1 < p < \infty$

(See [Ricard, Mei 2017], or [Gonzalez-Perez, Parcet, Xia 2022])

Definition: Let $G_0 \leq G$, and $m : G \to \mathbf{C}$ a function. We say that m is G_0-equivariant if there exist a character $\chi : G_0 \to \mathbf{C}$ such that

$$m(g_0 g) = \chi(g_0) m(g), \qquad g_0 \in G_0,\ g \in G.$$

Equivariant version of Cotlar [GPPX]

If m is bounded, real-valued, G_0-equivariant symbol, and

$$\big(m(g^{-1}) - m(h)\big)\big(m(gh) - m(g)\big) = 0, \quad g \in G \setminus G_0,\ h \in G$$

then T_m is L_p-bounded for $1 < p < \infty$.

An application of Ball Theorem

On $\mathrm{SL}_2(\mathbf{C})$ there are no idempotent Fourier multipliers!

Recall that

$$\mathrm{SL}_2(\mathbf{C}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; : \; \det A = 1, \quad a, b, c, d \in \mathbf{C} \right\}$$

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The lattice $\mathrm{SL}_2(\mathbf{Z}[i])$ has a L_p-bdd idempotent Fourier multiplier, $1 < p < \infty$.

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- There is no *natural lifting* method from lattices to $\mathrm{SL}_2(\mathbf{C})$
- Recently some *transference* methods has been studied [Wang, Xia, Yao 2025]...

Usually, **lattice** and **Lie Group** behave similarly, so...

$$\boxed{\textbf{Question: } \text{is } \mathrm{SL}_2(\mathbf{Z}[i]) \leq \mathrm{SL}_2(\mathbf{C}) \text{ an exception?}}$$

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The proof is very **arithmetic** $\xrightarrow{\text{conjecture}}$ it may work with **Bianchi groups**

$$B_n = \mathrm{SL}_2(O_n), \quad \text{where} \quad O_n = \mathbf{Z}[\sqrt{-n}] \text{ or } \mathbf{Z}\left[\frac{1+\sqrt{-n}}{2}\right]$$

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$$\boxed{\textbf{Obstruction: } \text{also, the proof is very arithmetic!!}}$$

It uses *very particular properties* of $\mathrm{SL}_2(\mathbf{Z}[i])$ to exploit $m\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \operatorname{sign} \operatorname{Re}\left(a\bar{c} + b\bar{d}\right)$

Theorem [PG 2024]

1. The multiplier m in [GP,P,X] does not satisfy Cotlar on Bianchi groups when $n > 1$,
2. However, there is an automorphism $\sigma \in \mathrm{Aut}\big(\mathrm{SL}_2(\mathbf{C})\big)$ such that

$$\tilde{m} = m \circ \sigma$$

   does satisfy Cotlar on every Bianchi group B_n with $n \neq 3$.

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Let's see how to prove (2) !

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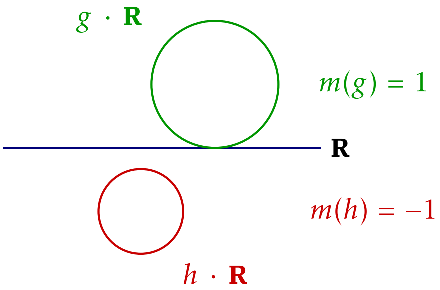
3. $\tilde{m}$ is defined by:



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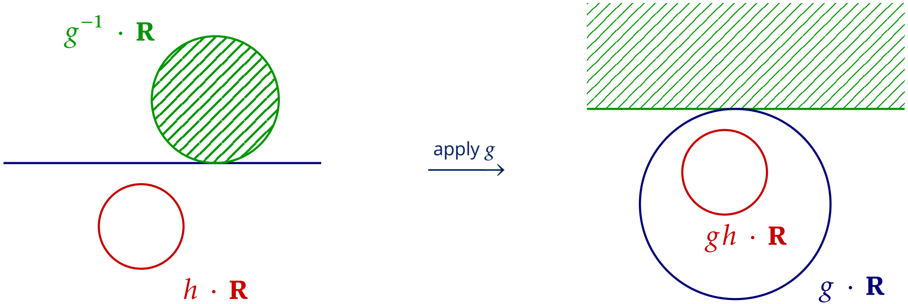

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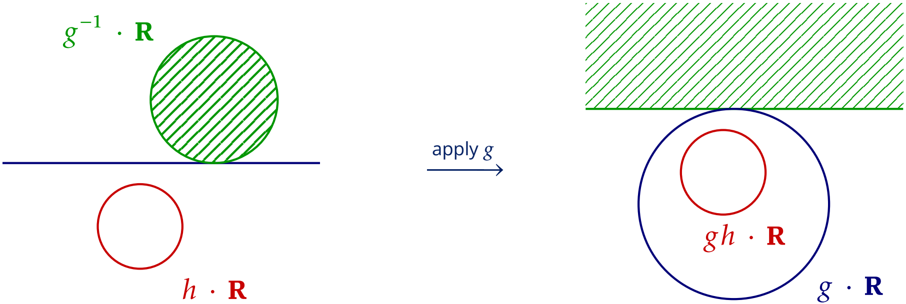

This means exactly that $m(gh) = m(g)$!

Some final notes

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2. *Conjecture:* similar multipliers may be found in lattices of

$$\mathrm{SO}(1,n) \cong \mathrm{Iso}(\mathbf{H}^n)$$

   (recall $\mathrm{SL}_2(\mathbf{C}) \cong SO(1,2) \cong \mathbf{H}^3$)

Thanks!