On the non-commutative Neveu decomposition and associated ergodic theorems

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Classical Setting

- $(\Omega, \mathcal{A}, \mu) \rightsquigarrow \sigma$-finite measure space.
- $T : \Omega \to \Omega$ a non-singular transformation. (i.e, mble and $\mu(T^{-1}(B)) = 0 \Leftrightarrow \mu(B) = 0$).

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- $(\Omega, \mathcal{A}, \mu, T) \rightsquigarrow$ **MPS** (measure preserving system) if $T : \Omega \to \Omega$ is a mble map and $\mu(T^{-1}(A)) = \mu(A), \forall A \in \mathcal{A}$.
- $\text{fix}(T) = \{A \in \mathcal{A} : \ T^{-1}(A) = A\}$
- $(\Omega, \mathcal{A}, \mu, T) \rightsquigarrow$ **ergodic** if $T^{-1}(A) = A \implies \mu(A) = 0$ or $\mu(A^c) = 0$.

Classical Ergodic theorems

Theorem 1 (**von Neumann and Birkhoff's Ergodic Theorem**)

$(\Omega, \mathcal{A}, \mu, T) \rightsquigarrow$ ***MPS***, *then the following holds;*

1. $\forall f \in L^2(X),\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \mathbb{E}(f \,|\, fix(T))$ *in* $\|\cdot\|_2$.

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2. $\forall f \in L^1(X)$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \mathbb{E}(f \mid fix(T))(x), \quad \mu\text{-a.e in } x,$$

and if $(\Omega, \mathcal{A}, \mu, T) \rightsquigarrow$ ***ergodic*** $\implies$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int f d\mu \quad \mu\text{-a.e in } x.$$

- For $1 \leq p < \infty$, and suppose $T : L^p(X, \mu) \to L^p(X, \mu)$ be a contraction
- Suppose $A_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(f)$ for $f \in L^p(X, \mu)$.
- **Does** $\lim_{n \to \infty} A_n(f)$ **exist** ?

von Neumann Mean Ergodic theorem

Theorem 2

Let $T : L^p(X,\mu) \to L^p(X,\mu)$ *be a contraction, suppose* $f \in L^p(X,\mu)$*, then following holds.*

- *Assume* $1 < p < \infty$*, then*

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- *For $p = 1$, $\lim_{n\to\infty} A_n(f)$ **may not exists** . But if T is Danford-Schwartcz class of operator i.e T is $L^1 - L^\infty$ contraction then*

$$\lim_{n\to\infty} A_n(f) \text{ exists in } \|\cdot\|_1 .$$

Pointwise and Krengel's stochastic ergodic theorems

Theorem 3

Let $T : L^p(X, \mu) \to L^p(X, \mu)$ be a positive contraction, suppose $f \in L^p(X, \mu)$, then following holds.

- *Assume $1 < p < \infty$, then*

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- *For* $p = 1$*, we have* ***stochastic convergence*** *(convergence in measur), i.e,*

$$\lim_{n\to\infty} A_n(f) \textit{ exists in measure } \mu.$$

Motivation: Invariant measures in classical setting

Stochastic ergodic theorem leads to a study of finding invariant measure and weakly wandering vectors.

- For a non-singular transformation T on a measure space $(\Omega, \mathcal{A}, \mu)$,
  Does $\exists$ a finite measure ν s.t. $\nu \circ T^{-1} = \nu$ and $\nu \sim \mu$?

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- A mble set W is **Weakly Wandering** if $\exists \{p_i\} \subseteq \mathbb{Z}_+$ s.t. $\{T^{-p_i}(W)\}$ are mutually disjoint.

Theorem 4 (Hajian-Kakutani, 1963)

The following are equivalent.

(i) $\exists$ *a finite measure* ν *s.t.* $\nu \circ T^{-1} = \nu$ *and* $\nu \sim \mu$.

(ii) *Does not exist any weakly wandering set of positive measure.*

Motivation: Invariant measures in classical setting

- Now note that $L^\infty(\Omega) \ni f \xrightarrow{T} f \circ T \in L^\infty(\Omega)$ is a positive isometric and identify $L^1(\Omega)^* = L^\infty(\Omega)$.
- Define the pre-dual map $\hat{T} : L^1(\Omega) \to L^1(\Omega)$ determined by

$$\int_\Omega \hat{T}(f) \cdot g d\mu = \int_\Omega f \cdot T(g) d\mu \; \forall f \in L^1(\Omega), \; g \in L^\infty(\Omega).$$

- $\hat{T}$ is a positive contraction and if T is invertible, it is given by $\hat{T}(f) = \frac{d\mu \circ T^{-1}}{d\mu} f \circ T^{-1}$ for all $f \in L^1(\Omega)$.

Motivation: Invariant measures in classical setting

- Let ν be a finite measure on $(\Omega, \mathcal{A})$ with $\nu \ll \mu$.
- $\exists f \in L^1(\Omega, \mu)$ such that $d\nu = f \cdot d\mu$.
- Then,

$$\hat{T}(f) = f \Leftrightarrow \nu \circ T^{-1} = \nu$$

Indeed, note that for all $g \in L^\infty(\Omega, \mu)$, we have,

$$\int_\Omega g \, d\nu \circ T^{-1} = \int_\Omega g \circ T d\nu = \int_\Omega g \circ T \cdot f \, d\mu = \int_\Omega g \cdot \hat{T}(f) \, d\mu.$$

Remark 5

Finding invariant measure for T is equivalent to finding fixed point of $\hat{T}$ in $L^1(\Omega, \mu)$.

Neveu Decomposition: Classical setting

Let $T : L^1(\Omega, \mu) \to L^1(\Omega, \mu)$ be a positive contraction.

Definition 6 (Krengel)

Let $T^* : L^\infty(\Omega, \mu) \to L^\infty(\Omega, \mu)$ *be the dual map of* T*. An element* $h \in L^\infty(\Omega, \mu)_+$ *is called*

- **weakly wandering** *if* $\exists$ *increasing sequence of integers* $0 = k_0 < k_1 < \ldots$ *such that* $\left\| \sum_0^\infty T^{*k_n} h \right\|_\infty < \infty$.

Remark 7

If h *is weakly wandering then* $\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{i=1}^n T^{*i} h \right\|_\infty = 0$.

Neveu Decomposition: classical setting

Theorem 8 (Krengel, 1985)

*$T \rightsquigarrow$ +ve **contraction** on $L^1(\Omega, \mu)$. Then $\exists$ disjoint sets $X_0, X_1 \in \mathcal{A}$ with $\Omega = X_0 \sqcup X_1$, uniquely determined upto null sets by the following properties.*

(i) $\exists$ *a* $g \in L^1(\Omega, \mu)_+$ *with* $Tg = g$ *and* $supp(g) = X_0$.

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(ii) *$\exists$ a $h \in L^\infty(\Omega, \mu)_+$ s.t h is weakly wandering and $supp(h) = X_1$.*

Stochastic Ergodic Theorem(Comm case)

Definition 9

A sequence of measurable functions $\{f_n\}$ *is said to converge stochastically to* f *if* $\forall\ \epsilon > 0$ *and* A *with* $\mu(A) < \infty$,

$$\lim_{n\to\infty} \mu(A \cap \{|f_n - f| > \epsilon\}) = 0.$$

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Theorem 10 (Stochastic Ergodic Theorem, Krengel, 1966)

$T \rightsquigarrow$ ***+ve contraction*** *on* $L^1(\Omega, \mu)$, *then* $\forall f \in L^1$,

$$\lim_{n\to\infty} A_n(f) \text{ exists stochastically },$$

and the limit is T-invariant and vanishes in X_1.

Neveu Decomposition(Non-Comm Case)

- $\mathcal{H} \rightsquigarrow$ **separble Hilbert space**. A unital $*$-subalgebra $M \subseteq \mathcal{B}(\mathcal{H})$ is **von Neumann algebra** (shortly vNa) if M is WOT closed ($\Leftrightarrow$ SOT closed $\Leftrightarrow M'' = M$).

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- Assume M finite vNa, $\exists$ a faithful, normal, tracial state τ on M. i.e, $\tau(xy) = \tau(yx), \forall x, y \in M$.
- For self-adjoint $x \in \mathcal{B}(\mathcal{H})$, s$(x)$ (support) is the smallest projection $e \in \mathcal{B}(\mathcal{H})$ such that $ex = xe = x$.

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- For self-adjoint $x \in \mathcal{B}(\mathcal{H})$, s$(x)$ (support) is the smallest projection $e \in \mathcal{B}(\mathcal{H})$ such that $ex = xe = x$.
- $M_* \subset M^*$ is the normed closed w-continuous/ normal linear functionals.
- Let $\phi \in M_*$ and $s(\phi)$ (support) is defined as

$$s(\phi) = 1 - \sup\{e \in \text{proj}(M) : \phi(e) = 0\}$$

Neveu Decomposition for group action

- $G \rightsquigarrow$ **2nd countable, locally compact, Hausdroff** (SLCH) group (semigroup) with a right Haar measure m.
- $G \rightsquigarrow$ **amenable** if $\exists$ a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of G of non-zero measure, s.t

$$\text{For all } g \in G, \quad \lim_{n\to\infty} \frac{m(K_n \Delta K_n g)}{m(K_n)} = 0.$$

- Consider a w-continuous group homomorphism $\alpha : G \to Aut(M)$ defined by $g \mapsto \alpha_g$, where $Aut(M)$ is the group of automorphisms of M.
- $(M, G, \alpha) \rightsquigarrow$ **covariant system**.

Neveu Decomposition for group action

- Write $\in \mathcal{P}_0(M) = \text{proj}(M) \setminus \{0\}$ and $\forall x \in M$ define,

$$A_n(x) := \frac{1}{m(K_n)} \int_{K_n} \alpha_g(x) dm(g).$$

- $x \in M_+$ is called weakly wandering if $\|A_n(x)\| \xrightarrow{n\to\infty} 0$

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Theorem 11 (Bikram, Saha)

Let M be a semi-finite von Neumann algebra with an f.n.s trace τ and (M, G, α) be a covariant system. Then $\exists\, e_1, e_2 \in proj(M)$ s.t. $e_1 + e_2 = 1$ and

- *$\exists$ a G-invariant state $\rho \in M_*$ s.t $s(\rho) = e_1$ and*
- *$\exists$ a weakly wandering operator $x_0 \in M$ with support $s(x_0) = e_2$.*

Further, $s(\rho)$ and $s(x_0)$ are unique.

Remark 12

In the non-coomutative setting Grabarnik and Katz (1995) took the first initiative and proved the above theorem for $\mathbb{Z}^d$-actions, $d \geq 1$.

Non-Commutative L^1

- Let (M, τ) be a tracial von Neumann algebra. and let $L^2(M, \tau)$ be the corresponding GNS representation and $M \subseteq \mathcal{B}(L^2(M, \tau))$. Write $\mathcal{H} = L^2(M, \tau)$.
- A (unbounded) operator x on $\mathcal{H}$ is affiliated to M if for all unitary $u \in M'$, we have $ux = xu$.
- $\tilde{M}$= set of all closed, densely defined operators affiliated with M.
- $\tilde{M}$, can be made a *-algebra with the natural actions.

Non-Commutative L^1

Theorem 13

$\{X \in \tilde{M} : \tau(|X|) < \infty\} = L^1(M, \tau)$

Non-Commutative L^1

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$$\{X \in \tilde{M} : \tau(|X|) < \infty\} = L^1(M, \tau)$$

- Given $X \in L^1(M, \tau)$, Define, $\|X\|_1 := \tau(|X|)$ and $(L^1(M, \tau), \|\cdot\|_1) \rightsquigarrow$ Banach space.
- Define, τ_a as a linear functional on M as $\tau_a(x) = \tau(ax)$ for all $x \in M$.

Theorem 14

Let (M, τ) be a tracial von Neumann algebra represented on $L^2(M, \tau)$. Then,

(i) *The map $a \mapsto \tau_a$ extends to a isomorphisms $L^1(M, \tau)$ onto M_* with*

$$\|\tau_a\| = \|a\|_1 .$$

(ii) *$L^1(M, \tau)$ is a Banach space with $L^1(M, \tau)^* = M$.*

Maximal type Inequality

$T : M \to M \rightsquigarrow$ **+ve, w-cont, contraction**, consider the dual operator $T^* : M^* \to M^*$. Write the ergodic averages corresponding to the operator T^* as

$$S_n(\mu) := \frac{1}{n}\sum_{k=0}^{n-1}(T^*)^k(\mu), \ \mu \in M^*.$$

Theorem 15 (Bikram, Saha)

Let M be a vNa with a f.n state ρ on M. Assume $T : M \to M \rightsquigarrow$ ***+ve contraction*** *s.t $T(1) \le 1$ and $\rho \circ T = \rho$. Let $\mu \in M_{*s}$ and $\epsilon > 0$. Then $\forall N \in \mathbb{N}, \exists \ e \in \mathcal{P}(M)$ such that $\rho(1-e) < \|\mu\|/\epsilon$ and*

$$|S_n(\mu)(x)| \le \epsilon\rho(x) \text{ for all } x \in (eMe)_+ \text{ and } n \in \{1, \ldots, N\}.$$

Maximal type Inequality

A compactly generated group G is said to be of **polynomial growth** if there exists $k > 0$ and $r \in \mathbb{N}$ such that $m(V^n) \le kn^r$ for all $n \in \mathbb{N}$.
Shortly write $(G, V) \rightsquigarrow$ LCHPG

Theorem 16 (Hong, Liao, Wang, 2021)

Suppose $(G, V) \rightsquigarrow$ LCHPG and let α be a strongly continuous action of G on an ordered Banach space E such that $\alpha_g(x) \ge 0$ for all $g \in G$ and $x \in E_+$. Define an operator $\mathcal{T}$ on E by

$$\mathcal{T}(x) = \frac{1}{m(V)} \int_V \alpha_g(x)dm(g), \textit{ for all } x \in E_+.$$

Then, there exists a constant c only depending on G such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g(x)dm(g) \le \frac{c}{2n^2} \sum_{k=1}^{2n^2} \mathcal{T}^k(x), \textit{ for all } x \in E_+.$$

Pointwise ergodic theorem: Predual action

Definition 17

A sequence of operators $\{X_n\}_{n\in\mathbb{N}} \subseteq L^1(M,\tau)$ *converges* ***bilaterally almost uniformly (b.a.u)*** *to* $X \in L^1(M,\tau)$ *if for all* $\delta > 0$ *there exists a projection* $e \in M$ *such that*

$$\tau(1-e) < \delta \text{ and } \|e(X_n - X)e\| \to 0.$$

Definition 18

Let $T : M \to M$ *be a positive contraction. Since* $L^1(M,\tau)^* = M$*, the predual map* $\hat{T} : L^1(M,\tau) \to L^1(M,\tau)$ *is defined by*

$$\tau(\hat{T}(X)y) = \tau(XT(y)) \; \forall X \in L^1(M,\tau), y \in M.$$

Pointwise ergodic theorem action

Let M be a von Neumann algebra with f.n. tracial state τ and $X \in L^1(M,\tau)$. When G is either LCHPG group or $\mathbb{Z}_+$;define

$$M_n(X) := \begin{cases} \frac{1}{m(V^n)} \int_{V^n} \hat{\alpha}_g(X) dm, \ n \in \mathbb{N} & \text{if } G = \text{LCHPG}, a \in \mathbb{N} \\ \frac{1}{n} \sum_{k=0}^{n-1} \hat{T}^k(X) & \text{if } G = \mathbb{Z}_+, \ n \in \mathbb{Z}_+, \end{cases}$$

Kernel : (M, G, T, ρ) is called kernel if $\exists$ a f.n. state $\rho \in M_*$ which is G-invariant.

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Theorem 19 (Bikram, Saha)

Let G be either a LCHPG group, $\mathbb{Z}_+$ and $(M, G, \alpha, \rho) \rightsquigarrow$ kernel . Suppose $\mu \in M_{*s}$ and $\epsilon > 0$. Then $\forall\ N \in \mathbb{N}\ \exists\ e \in$ proj(M) s.t $\rho(1-e) < c \left\|\mu\right\| / \epsilon$ and

$$|M_a(\mu)(x)| \leq \epsilon\rho(x) \text{ for all } x \in (eMe)_+ \text{ and } a \in \{1, \ldots, N\}.$$

Banach Principle

- For $v \in M_{*+}$, suppose $\overline{v} = \|\cdot\|_1 \text{-} \lim_{n\to\infty} A_n(v)$. Then consider $W = \{v - M_k(v) + \overline{v} : \ k \in \mathbb{N}, v \in M_{*+}, \ \text{with } v \leq \lambda\rho, \ \text{for some } \lambda > 0\}$. Then $W - W$ is dense in M_{*s}.
- Banach Principle type result:

Theorem 1 (Bikram + Saha, 2025)

*$(M, G, \alpha, \rho) \rightsquigarrow$ **karnel**, suppose $M \rightsquigarrow$ **finite vNa with a f.n trace τ**.*

Then, the following set

$$\mathcal{C} = \{\phi \in M_{*s} : M_n(\phi) \text{ converges in b.a.u}\}$$

is closed in M_{*s}.

- $W - W \subseteq \mathcal{C}$.

Theorem 20 (Bikram, Saha)

Let (M, G, T, ρ) *be a kernel and* τ *be a f.n tracial state on* M. *Then* $\forall\ Y \in L^1(M, \tau)$, $\exists\ \overline{Y} \in L^1(M, \tau)$ *s.t* $M_a(Y)$ *converges to* $\overline{Y}$ *bilaterally almost uniformly.*

Stochastic Ergodic theorem

Fix a covariant system (M, G, α), where G is a LCHPG group with a fixed compact symmetric generating set V.

- Let e_1, e_2 be the projection obtained Neveu Deco. Then note that $\alpha_g(e_i) = e_i, \forall g \in G$ and $i = 1, 2$. Write $M_{e_i} = e_i M e_i$ and then (M_{e_i}, G, α) for $i = 1, 2$, becomes a covariant system.
- $\hat{\alpha}_g(e_i X e_i) = e_i \hat{\alpha}_g(X) e_i$ for all $X \in L^1(M, \tau)$ and for all $g \in G$. As a consequence, the ergodic averages satisfy

$$e_i A_n(X) e_i = A_n(e_i X e_i) \text{ for all } X \in L^1(M, \tau), n \in \mathbb{N} \text{ and } i = 1, 2.$$

Stochastic Ergodic theorem

Defn: A sequence $\{X_n\} \subset L^1(M, \tau)$ is said to converge to $X \in L^1(M, \tau)$ **stochastically or in measure** if $\forall \epsilon, \delta > 0$ there exists $n_0 \in \mathbb{N}$ and a sequence of projections $\{e_n\}_{n \in \mathbb{N}} \subset M$ such that $\|e_n(X_n - X)e_n\| < \epsilon$ and $\tau(1 - e_n) < \delta$ for $n \geq n_0$.

Stochastic Ergodic theorem

Theorem 21 (Bikram, Saha)

Let $(M, G, \alpha,) \rightsquigarrow$ ***covariant system****. Then* $\exists\, e_1, e_2 \in$ *proj*(M) *(as in Neveu Deco.) s.t;*

1. $\forall B \in L^1(e_1Me_1, \tau_{e_1}),\ \exists \bar{B} \in L^1(e_1Me_1, \tau_{e_1})$ *s.t*
    - $A_n(B)$ *converges b.a.u to* $\bar{B}$*.*
    - *In particular,* $A_n(B)$ *converges in measure to* $\bar{B}$*.*
2. $\forall B \in L^1(e_2Me_2, \tau_{e_2})$, $A_n(B)$ *converges in measure to* 0*.*

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Stochastic Ergodic theorem

Theorem 22 (Bikram, Saha)

Let $(M, G, \alpha,) \rightsquigarrow$ ***covariant system*** . *Then*
$\forall X \in L^1(M, \tau),\ \ \exists\ \ Z \in L^1(M, \tau)$ *s.t* $A_n(X)$ *converges to* Z *in measure.*

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Stochastic Ergodic theorem

Theorem 22 (Bikram, Saha)

Let $(M, G, \alpha,) \rightsquigarrow$ ***covariant system*** . *Then*
$\forall X \in L^1(M, \tau),\ \ \exists\ Z \in L^1(M, \tau)$ *s.t* $A_n(X)$ *converges to* Z *in measure.*

Remark 23

For the semigroup to prove the above theorem, we need to assume that the maps to be Lamperti operators.

Thanks