Khintchine Inequalities and Z_2 Sets Revisited
Commutative v.s Noncommutative

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Joint work with ChianYeong Chuah and Zhenchuan Liu

Weighted Sum of i. i. d. Random Variables

Tossing a Fair Coin:

ε_k: the outcome of the k-th tossing, $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$;

- Let

$$X_n = \sum_{k=1}^{n} c_k \varepsilon_k;$$

Basic Questions

- Estimate $\mathbb{E}|X_n|^p$ for n large?
- When does $\sum_{k=1}^{\infty} c_k \varepsilon_k$ converge?

$$EX_n = 0, \quad \mathbb{E}X_n^{2j+1} = 0.$$

$$\mathbb{E}X_n^2 = \sum_{k,j=1}^{n} c_k c_j \mathbb{E}\varepsilon_k \varepsilon_j = \sum_{k=1}^{n} c_k c_k \mathbb{E}\varepsilon_k^2 = \sum_{k=1}^{n} c_k^2.$$

The Khintchine Inequality

ε_k: i. i. d. random variables with the Rademacher distribution
$P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$.

For $0 < p < \infty$, $c_k \in \mathbb{R}/\mathbb{C}$,

$$A_p \left(\sum_{k=1}^{n} |c_k|^2 \right)^{\frac{p}{2}} \leqslant \mathbb{E} \left| \sum_{k=1}^{n} \varepsilon_k c_k \right|^p \leqslant B_p \left(\sum_{k=1}^{n} |c_k|^2 \right)^{\frac{p}{2}}.$$

Aleksandr Yakovlevich Khinchin (1894–1959)
Sergei Natanovich Bernstein (1880-1968)

The Khintchine Inequality

ε_k: i. i. d. random variables with the Rademacher distribution $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$.

For $0 < p < \infty$, $c_k \in \mathbb{C}$,

$$\left(\mathbb{E}\left|\sum_{k=1}^{n} \varepsilon_k c_k\right|^p\right)^{\frac{1}{p}} \simeq^{K_p} \left(\sum_{k=1}^{n} |c_k|^2\right)^{\frac{1}{2}}.$$

$K_1 = \sqrt{2}$ (Szarek 1976)

ε_k can be replaced by

- Independent standard Gaussian(=normal) random variables.
- Complex variables $e^{i2^k t} \in L^p(\mathbb{T})$.

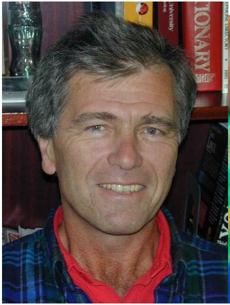
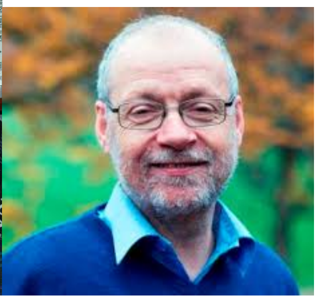
Connections

- Independent centered variables: Burkholder-Gundy inequality;
- Voiculescu's free independent random variables.
- Λ_p-set: Zygmund, Rudin, Erdös, Bourgain,
- Structure of Metric Spaces: Mendel/Naor; Ivanisvili/van Hanel/Volberg
- Kahane's Inequality; Random Matrices; Matrix-valued c_k.

The Khintchine inequalities for (matrix) operator- c_k

Assume $0 < p < \infty$, then for $c_k \in M_N$, we have

$$\left(\mathbb{E}\mathrm{tr}\left|\sum_{k=1}^{n} c_k \otimes \varepsilon_k\right|^p\right)^{\frac{1}{p}} \approx^{K_p} \left\|(c_k)_k\right\|_{S_p(\ell_2)}.$$

Lust-Piquart 1986; Lust-Piquart/Gilles Pisier in 1991; Haagerup/Musat 2017; Pisier/Ricard 2017; Cadilhac 2019.

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for $p \geqslant 2$,

$$\Big\|(c_k)_k\Big\|_{S^p(\ell^2)} := \max\{\mathrm{tr}\Big(\sum_{k=1}^{n} c_k c_k^*\Big)^{\frac{p}{2}}, \mathrm{tr}\Big(\sum_{k=1}^{n} c_k^* c_k\Big)^{\frac{p}{2}}\}.$$

for $0 < p < 2$,

$$\Big\|(c_k)_k\|_{S^p(\ell^2)} := \inf_{c_k = a_k + b_k} \{\mathrm{tr}\Big(\sum_{k=1}^{n} a_k a_k^*\Big)^{\frac{p}{2}}, \mathrm{tr}\Big(\sum_{k=1}^{n} b_k^* b_k\Big)^{\frac{p}{2}}\}.$$

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- $K_p \simeq \sqrt{p}, p \to \infty$;
- $\|\cdot\|_{M_N} \leqslant N^{\frac{1}{p}} \|\cdot\|_{S_N^p} \leqslant \|\cdot\|_{M_N}$

For $p = \infty$, $c_k \in M_N$,

$$\mathbb{E}\left\|\sum_{i=k}^{n} c_k \otimes \varepsilon_k\right\|_{M_N} \lesssim \sqrt{\log N} \max\{\left\|\sum_{k=1}^{n} c_k c_k^*\right\|_{M_N}^{\frac{1}{2}}, \left\|\sum_{k=1}^{n} c_k^* c_k\right\|_{M_N}^{\frac{1}{2}}\}.$$

Random Matrices, Concentration inequality, $\cdots$

The Khintchine inequalities for (matrix) operator- c_k

Assume $0 < p < \infty$, then for $c_k \in M_N$, we have

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Haaggerup and Musat 2017

- $K_1 = \sqrt{2}$ for $\varepsilon_k = e^{i2^k t} \in L^1(\mathbb{T})$.
- $K_1 \leqslant \sqrt{3}$ for ε_k with Rademarcher's distribution $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}$.

Optimal Khintchine inequalities and Z_2 sets

Haaggerup and Musat 2017 For $A = \{2^j, j \in \mathbb{N}\}$,

$$\|(c_k)_k\|_{S^1(\ell_2)} \leqslant K_1(A) \int_0^1 tr|\sum_{k\in A} c_k \otimes e^{i2\pi k\theta}|d\theta \qquad (Kh1)$$

Here $A = \{2^j, j \in \mathbb{N}\}$ is a Z_2 set in the sense that.

$$Z_2(A) = \sup_{n\in\mathbb{N}}\{(a,b) \in A \times A; a - b = n\} = 1 < \infty.$$

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$$Z_2(A) < \infty \Rightarrow K_1(A) \leqslant \sqrt{Z_2(A) + 1},$$

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$$Z_2(A) < \infty \Rightarrow K_1(A) \leqslant \sqrt{Z_2(A) + 1},$$

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$$K_1(A) = \sqrt{2} \Rightarrow Z_2(A) \leqslant 6;$$

Optimal Khintchine inequalities and Z_2 sets

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$$K_1(A) = \sqrt{2} \Rightarrow Z_2(A) \leqslant 6; \quad K_1(A) < \sqrt{2} + \delta \Rightarrow Z_2(A) < \infty.$$

Commutative v.s. noncommutative

Recall for $A \subset \mathbb{Z}, c_k \in M_N$, $Z_2(A) = \sup_{n \in \mathbb{N}} \{(a,b) \in A \times A; a - b = n\}$

$$\left(tr \int_0^1 \left|\sum_{k \in A} c_k \otimes e^{i2\pi kt}\right|^p dt\right)^{\frac{1}{p}} \simeq^{K_p(A)} \|(c_k)_k\|_{S^p(\ell_2)} \quad (Khp)$$

Zygmund,Rudin...

$$Z_2(A) < \infty \Rightarrow K_4(A) \leqslant (Z_2(A) + 1)^{\frac{1}{4}}.$$

Pisier-Ricard 2017

$$K_p(A) < \infty \Rightarrow K_q(A) < \infty, \forall 0 < q < p.$$

Unclear, whether for matrix(operator)-valued c_k,

$$K_1(A) \leqslant (K_4(A))^2?$$

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Ricard 2017 The Mazur map $x \mapsto x|x|^{\frac{p-q}{q}}$ is α-Hölder from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$.

The Khintchine inequality for systems

$\mathcal{M}$: von Neumann algebra with a finite trace τ.
$L^p(\mathcal{M})$: associated noncommutative L^p-space
$\mathcal{A} = \{x_k \in L_2(\mathcal{M}), \tau(x_k^* x_j) = \delta_{kj}\}$
$\tau(|x_k|^2|x_j|^2) = 1, \forall x_k, x_j \in \mathcal{A}$

Chuah-Liu-M, 2025

$$Z_2(\mathcal{A}) < \infty \Rightarrow K_1(\mathcal{A}) \leqslant \sqrt{Z_2(\mathcal{A}) + 1},$$

For a class of abelian group von Neumann algebra,

$$K_1(\mathcal{A}) < \sqrt{2} + \delta \Rightarrow Z_2(\mathcal{A}) < \infty.$$

Here $K_1(\mathcal{A})$ is the best constant s.t.

$$\|(c_k)_k\|_{S^1(\ell_2)} \leqslant K_1(\mathcal{A}) tr \otimes \tau |\sum_{x_k \in \mathcal{A}} c_k \otimes x_k| \qquad (Kh1)$$

Example

$\mathcal{M} = L^\infty([0,1])$, $\tau = \int$
$L^p(\mathcal{M}) = L^p([0,1])$
$\mathcal{A} = \{\varepsilon_k = sign(2^k \pi t), t \in [0,1]\}$

$$Z_2(\mathcal{A}) = 2; K_1(\mathcal{A}) \leqslant \sqrt{3}.$$

Open question: $K_1(\mathcal{A}) = \sqrt{2}$?

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Example

$\mathcal{M} = L^\infty(\mathbb{T}), \tau = \int, \mathcal{A} = \{e^{i2\pi 2^k t}; k \in \mathbb{N}\}$

$$Z_2(\mathcal{A}) = 1; K_1(\mathcal{A}) = \sqrt{2}.$$

$\mathcal{M} = L^\infty(\mathbb{T}), \tau = \int, \mathcal{A} = \{e^{i2\pi 2^k 3^j t}, k, j \in \mathbb{N}\},$

$$Z_2(\mathcal{A}) = 3, K_1(\mathcal{A}) \leqslant 2.$$

G : discrete group.
$\mathcal{M}$=Group von Neuman algebra G
$\tau(\lambda_g) = \delta_e(g)$
$\mathcal{A} = \{\lambda_{g_k}, k \in \mathbb{N}; \frac{|g_{k+1}|}{|g_k|} > 2\}$

$$Z_2(\mathcal{A}) = 1; K_1(\mathcal{A}) = \sqrt{2}.$$

Example: Quantumn Rademacher Sequences

σ_i = Pauli matrices $\in M_2, i = 1, 2, 3$

$$\varepsilon_{i,k} = id_2 \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes id_2 \in M_{2^N}, 1 \leqslant k \leqslant N.$$

$\mathcal{A} = \{\varepsilon_{i,k}; 1 \leqslant k \leqslant N, 1 \leqslant i \leqslant 3\} \subset M_{2^N}$

$$Z_2(\mathcal{A}) = 2 \Rightarrow K_1(\mathcal{A}) \leqslant \sqrt{3}.$$

Khintchine inequality for systems

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$\mathcal{A} = \{x_k \in L_2(\mathcal{M}), \tau(x_k^* x_j) = \delta_{kj}\}$

$$N_1(\mathcal{A}) = \sup_{x \in \mathcal{A}} \tau(|x|^4), \quad N_2(\mathcal{A}) = \{\tau(|x|^2|y|^2) : x, y \in \mathcal{A} \text{ and } x \neq y\}$$

$$Z_{2,2}(\mathcal{A}) = \sup_{w,x \in \mathcal{A}, w \neq x} \{ \sum_{y,z \in \mathcal{A}} |\tau(w^* x y^* z)|, \sum_{y,z \in \mathcal{A}} |\tau(w x^* y z^*)| \},$$

$$Z_{2,1}(\mathcal{A}) = \sup_{x \in \mathcal{A}} \{ \sum_{y,z \in \mathcal{A}, y \neq z} |\tau(|x|^2 y^* z)|, \sum_{y,z \in \mathcal{A}, y \neq z} |\tau(|x^*|^2 y z^*)| \}.$$

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$$K_1(\mathcal{A}) \leqslant max\{\sqrt{N_1(\mathcal{A}) + Z_{2,1}(\mathcal{A})}; \sqrt{N_2(\mathcal{A}) + Z_{2,2}(\mathcal{A})}\}$$

Khintchine inequality for systems

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Chuah-Liu-M, 2025

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Here $K_1(\mathcal{A})$ is the best constant s.t.
$\|(c_k)_k\|_{S^1(\ell_2)} \leqslant K_1(\mathcal{A}) tr \otimes \tau |\sum_{x_k \in \mathcal{A}} c_k \otimes x_k| \qquad (Kh1)$

Example

$\mathcal{A} = \{(\gamma_n)_{n=1}^{\infty}$: independent, standard, complex-valued Gaussian$\}$. Then $N_1(\mathcal{A}) = 2, N_2(\mathcal{A}) = Z_{2,2}(\mathcal{A}) = 1, Z_{2,1}(\mathcal{A}) = 0$.

$$K_1(\mathcal{A}) = \sqrt{2}.$$

$\mathcal{A} = \{(\gamma_n)_{n=1}^{\infty}$: independent, standard, real-valued Gaussian $\}$. Then $N_1(\mathcal{A}) = 3, N_2(\mathcal{A}) = Z_{2,2}(\mathcal{A}) = 1, Z_{2,1}(\mathcal{A}) = 0$.

$$K_1(\mathcal{A}) \leqslant \sqrt{3}.$$

σ_i = Pauli matrices $\in M_2, i = 1, 2, 3$
$\mathcal{A} = \{\varepsilon_{i,k} = id_2 \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes id_2; 1 \leqslant k \leqslant N, 1 \leqslant i \leqslant 3\} \subset M_{2^N}$

$$N_1(\mathcal{A}) = N_2(\mathcal{A}) = 1, Z_{2,2}(\mathcal{A}) = 2, Z_{2,1}(\mathcal{A}) = 0 \Rightarrow K_1(\mathcal{A}) \leqslant \sqrt{3}.$$

$K_1(\mathcal{A}) \leqslant max\{\sqrt{N_1(\mathcal{A}) + Z_{2,1}(\mathcal{A})}, \sqrt{N_2(\mathcal{A}) + Z_{2,2}(\mathcal{A})}\}$

$$\|\tau|\sum_{k\in A} c_k \otimes x_k|^4\|^{\frac{1}{4}} \leqslant \mathcal{K}_4(\mathcal{A}) \max\{\|\sum|c_k|^4\|^{\frac{1}{4}}, \|\sum|c_k^*|^4\|^{\frac{1}{4}}\} \quad (\mathcal{K})$$

Theorem Chuah-Liu-M 2025 (following Haagerup/Musat 2017)

$$\mathcal{K}_4^4(\mathcal{A}) \leqslant max\{N_1(\mathcal{A}) + Z_{2,1}(\mathcal{A}), N_2(\mathcal{A}) + Z_{2,2}(\mathcal{A})\}$$

$$K_1(\mathcal{A}) \leqslant (\mathcal{K}_4(\mathcal{A}))^2 = max\{\sqrt{N_1(\mathcal{A}) + Z_{2,1}(\mathcal{A})}, \sqrt{N_2(\mathcal{A}) + Z_{2,2}(\mathcal{A})}\}.$$

Proof of $K_1(A) \leqslant \sqrt{2} \Rightarrow Z_2(A) \leqslant 6$

Key Lemma (Haagerup-Itoh,1995) For any $n \in \mathbb{N}$, there exist $2n+1$ partial isometries $a_1, \ldots, a_{2n+1} \in M_d(\mathbb{C})$ with $d = \binom{2n+1}{n}$ such that

1) $tr(a_i^* a_i) = \binom{2n}{n}$.
2) $\sum_{i=1}^{2n+1} a_i^* a_i = \sum_{i=1}^{2n+1} a_i a_i^* = (n+1) I_d$.
3) For any $(g_k)_{k=1}^{2n+1} \subset \mathbb{C}$ with $\sum_{k=1}^{2n+1} |g_k|^2 = 1$, the operator $b = \sum_{k=1}^{2n+1} g_k a_k$ is a partial isometry with $tr(b^* b) = \binom{2n}{n}$.

For $n = 2, d = 10$

$$\begin{aligned} a_1 &= E_{6,1} + E_{5,2} + E_{4,3} + E_{3,4} + E_{2,5} + E_{1,6}, \\ a_2 &= -E_{9,1} - E_{8,2} - E_{7,3} + E_{3,7} + E_{2,8} + E_{1,9}, \\ a_3 &= -E_{8,4} - E_{7,5} - E_{5,7} - E_{4,8} + E_{1,10} + E_{10,1}, \\ a_4 &= E_{10,2} + E_{9,4} + E_{6,7} - E_{7,6} - E_{4,9} - E_{2,10}, \\ a_5 &= E_{10,3} + E_{9,5} + E_{8,6} + E_{6,8} + E_{5,9} + E_{3,10}. \end{aligned}$$

satisfying the 1),2) and 3) where $E_{i,j}$ are the coefficient matrices.

Proof: Cases study

Let $A \subseteq \mathbb{Z}$. Suppose that any of the following holds

1. A contains an arithmetic sequence of length 5.
2. there exist distinct $k_1, k_2, ..., k_{10} \in A$ such that $k_2 - k_1 = k_4 - k_3 = ... = k_{10} - k_9$
3. there exist distinct $k_1, k_2, ..., k_9 \in A$ such that (k_1, k_2, k_3), (k_4, k_5, k_6) and (k_7, k_8, k_9) are arithmetic progressions of length 3 with same common difference.

then

$$K_1(A) > \sqrt{2}.$$

Theorem (Chuah-Liu-M 25):

$$K_1(A) = \sqrt{2} \Rightarrow Z_2(A) \leqslant 6$$

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Theorem (Chuah-Liu-M 25):

$$K_1(A) = \sqrt{2} \Rightarrow Z_2(A) \leqslant 6 \Rightarrow K_4(A) \leqslant 7^{\frac{1}{4}}.$$

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