Tensor products and R-matrices for quantum toroidal algebras

Duncan Laurie

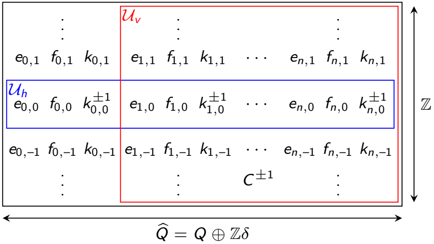
University of Edinburgh

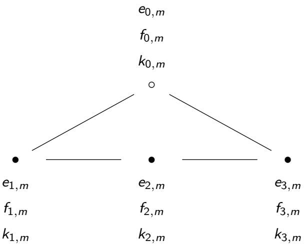
Further details: `arXiv:2503.08839`

Quantum toroidal algebras $U_q(\mathfrak{g}_{\mathrm{tor}})$

Definition: the *quantum affinizations* of quantum affine algebras

Construction: index Drinfeld's loop-style realization for $U_q(\hat{\mathfrak{g}})$ over *affine* Dynkin diagram





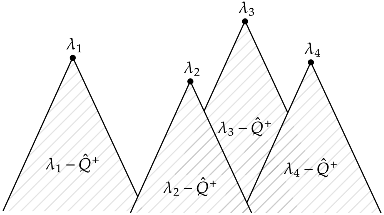
Topological coproduct: $\Delta_u : U_q(\mathfrak{g}_{\mathrm{tor}}) \to U_q(\mathfrak{g}_{\mathrm{tor}}) \widehat{\otimes} U_q(\mathfrak{g}_{\mathrm{tor}})$, eg.

$$\Delta_u(e_{i,0}) = e_{i,0} \otimes 1 + \sum_{\ell \geq 0} (C^{-\ell} k_{i,\ell} \otimes e_{i,-\ell})\, u^{\ell}$$

The module category $\widehat{\mathcal{O}}$

Definition: category $\widehat{\mathcal{O}}$ of $U_q(\mathfrak{g}_{\mathrm{tor}})$-modules whose restrictions to $\mathcal{U}_h$ lie inside its category $\mathcal{O}$, ie.

1. all $e_{i,0}$ and $f_{i,0}$ act locally nilpotently
2. decompose as direct sum of finite-dimensional weight spaces w.r.t $\langle k_{0,0}^{\pm 1}, \dots, k_{n,0}^{\pm 1} \rangle$
3. weights lie inside finite union of cones



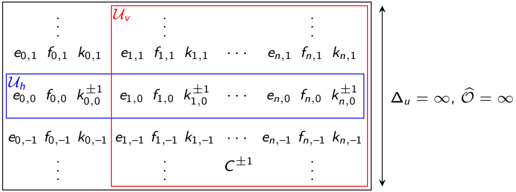
Motto: · principal category $\widehat{\mathcal{O}}$ of integrable $U_q(\mathfrak{g}_{\mathrm{tor}})$ representations

· toroidal analogue of finite-dimensional modules for quantum affine algebras

Problem: $\Delta_u \overset{\text{nope!}}{\rightsquigarrow} \otimes$ on $\widehat{\mathcal{O}}$ ☹

Problem

Motto: both Δ_u and repns in $\widehat{\mathcal{O}}$ are *'vertically infinite'*

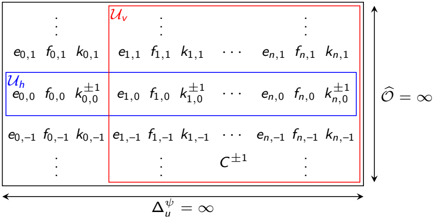


Example: · $\Delta_u(e_{i,0}) = e_{i,0} \otimes 1 + \sum_{\ell \geq 0} (C^{-\ell} k_{i,\ell} \otimes e_{i,-\ell})\, u^\ell$

· for $v \in V \in \widehat{\mathcal{O}}$, common that $k_{i,m} \cdot v \neq 0$ and $e_{i,m} \cdot v \neq 0$ for all $m \in \mathbb{Z}$

$$\therefore \ e_{i,0} \cdot (v \otimes w) = (e_{i,0} \cdot v) \otimes w + \sum_{\ell \geq 0} \underbrace{(k_{i,\ell} \cdot v)}_{\neq 0} \otimes \underbrace{(e_{i,-\ell} \cdot w)}_{\neq 0} u^\ell = \ ???$$

Solution



Game plan [L24a,L24b,L25][1]**:**

1. (extended) double affine braid group action $\ddot{\mathcal{B}} \curvearrowright U_q(\mathfrak{g}_{\mathrm{tor}})$
2. duality involution $\mathfrak{t}$ of $\ddot{\mathcal{B}}$ $\overset{\text{pass across action}}{\rightsquigarrow}$ anti-involution[2] ψ of $U_q(\mathfrak{g}_{\mathrm{tor}})$ swapping $\mathcal{U}_h \leftrightarrow \mathcal{U}_v$
3. *horizontally infinite* topological coproduct $\Delta_u^\psi = (\psi \otimes \psi) \circ \Delta_u \circ \psi$

[1] Building on works by Miki for $U_q(\mathfrak{sl}_{n+1,\mathrm{tor}})$

[2] $\psi \implies \widetilde{GL_2(\mathbb{Z})} \ltimes \ddot{\mathcal{B}} \curvearrowright U_q(\mathfrak{g}_{\mathrm{tor}})$ in all types, containing generalisation of celebrated Miki automorphism

Tensor product

Theorem [L25]: $\Delta_u^\psi \overset{\text{yes!}}{\rightsquigarrow} \otimes$ on $\widehat{\mathcal{O}} \rightsquigarrow K(\widehat{\mathcal{O}})$ is *ring* ☺ (separately $\forall u \in \mathbb{C}^\times$)

Resembles Drinfeld–Jimbo coproduct and tensor product on affine level [3] *...*

- $V \otimes W \cong V \otimes_{\mathrm{DJ}} W$ as representations of $\mathcal{U}_v$
- $\otimes$ of irreducibles is generically irreducible w.r.t. spectral parameter (cf. [Chari–Pressley])

Various nice properties ...

- compatible with Drinfeld polys that classify irreducibles: $\mathcal{P}_i(V \otimes W) = \mathcal{P}_i(V) \cdot \mathcal{P}_i(W)$
- compatible with q-characters[4]: $\chi_q(V \otimes W) = \chi_q(V) \cdot \chi_q(W)$

  upgrades $\chi_q : K(\widehat{\mathcal{O}}) \hookrightarrow \mathcal{Y}$ to *ring* homomorphism

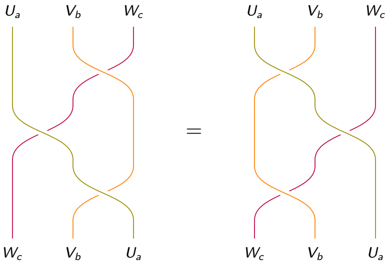
[3] compare: $\otimes$ for $Y_\hbar(\hat{\mathfrak{g}})$ due to [Guay–Nakajima–Wendlandt '18], though constructed rather differently!

[4] $\therefore$ my tensor product $\otimes$ and 'orthogonal' *fusion product* $*$ [Hernandez '07] become equal on level of $K(\widehat{\mathcal{O}})$

Theorem [L25][5]**:** There exist meromorphic $\mathcal{R}(x) : V \otimes W \to W \otimes V$ such that

> $U_q(\mathfrak{g}_{\mathrm{tor}})$*-module homomorphism except at poles;* *isomorphism generically;* *YBE satisfied*

which equip $\widehat{\mathcal{O}}$ with a meromorphic braiding.



[5] V, W are $\bigoplus \bigotimes$ irreducibles for now, but will extend