Representation theory of non-factorizable ribbon Hopf algebras

Maksymilian Manko, University of Zurich
arXiv:2503.19532 (joint with Q.Faes), arXiv:2510.24263

Edinburgh, 2025

Motivation - 4-dimensional topology

- **(Non-)semisimple 3d TQFTs and related 3-manifold invariants require factorizable ribbon Hopf algebras/categories**
  Reshetikhin and Turaev (1991), Kerler and Lyubashenko (2001), Costantino, Geer, and Patureau-Mirand (2014)
- **Non-semisimple 4d TQFTs and related 4-manifold invariants require non-factorizabile unimodular ribbon categories**

  Kerler-Lyubashenko (KL) functors Beliakova and de Renzi (2023), Bobtcheva and Piergallini (2006) - give an invariant of 4-dimensional 2-handlebodies up to 2-equivalence (Kirby moves involving only 1-, 2-handles)

  If H is factorizable ribbon Hopf algebras, the KL functor of a 4-dimensional 2-handlebody W, $J_H(W)$ depends only on 3-dimensional ∂W.

  4d TQFT Costantino, Geer, Haïoun, and Patureau-Mirand (2023) on 4-manifolds up to diffeomorphisms

  If $\mathcal{C}$ is a modular ribbon category, 4d TQFT functor associated to a 4-manifold W, $J_{\mathcal{C}}(W)$ depends only on 3-dimensional ∂W.

Overview

1. Preliminaries and factorizability
2. Strongly non-factorizable Hopf algebras
3. Nenciu biproducts with $u_q\mathfrak{sl}_2$
4. Representation theory and their symmetric centres

Preliminaries

Conventions

All Hopf algebras considered are going to be over $\mathbb{C}$, finite dimensional. They are also usually non-semisimple. We use the usual notation $\mathbf{1}, \Delta, S$.

Definition

Let H be a Hopf algebra carrying a (left) integral $\lambda \in H^*$ and a (left) cointegral $\Lambda \in H$. It is called *unimodular* if Λ is two-sided, that is $S(\Lambda) = \Lambda$. We can normalise $\lambda(\Lambda) = \mathrm{id}_{\mathbb{1}}$. Equivalently in H-mod, $P_{\mathbb{1}} \cong P_{\mathbb{1}}^*$.

Definition

Let H be a Hopf algebra and $R \in H \otimes H$. The pair (H, R) is *quasitriangular* if

(QT1) $\Delta(R') \otimes R'' = R_1' \otimes R_2' \otimes R_1'' R_2''$

(QT2) $\epsilon(R')R'' = \mathbf{1}$

(QT3) $R' \otimes \Delta^{cop}(R'') = R_1' R_2' \otimes R_1'' \otimes R_2''$

(QT4) $\epsilon(R'')R' = \mathbf{1}$

(QT5) $\Delta^{cop}(h)R = R\Delta(h), \forall h \in H$

Then R has an inverse denoted R^{-1}. $M := R_{21}R$ is the monodromy matrix
If $M = \mathbf{1} \otimes \mathbf{1}$, H is called *triangular*.

Factorizability I

Let (H, R) finite-dimensional quasitriangular Hopf algebra. Then H-mod is a braided category. Denote the braiding by $c_{W,V} : W \otimes V \to V \otimes W$ for $W, V \in H$-mod

Definition

Define the *symmetric centre* to be the full subcategory of *transparent objects*

$$\mathcal{Z}_{(2)}(H\text{-mod}) = \{W \in H\text{-mod} \,|\, c_{W,V} \circ c_{V,W} = \mathrm{Id}_{W\otimes V}, \forall V \in H\text{-mod}\}$$

Definition (Shimizu (2016))

The quasitriangular Hopf algebra (H, R) (and its H-mod) is called *factorizable* if any of the two equivalent conditions are satisified

1. the Drinfeld map

$$H^* \to H \qquad h^* \mapsto h^*(M')M''$$

   is an isomorphism of vector spaces,
2. symmetric centre $\mathcal{Z}_{(2)}(H\text{-mod})$ is trivial.

Proposition (Kerler and Lyubashenko (2001))

If H is a factorizable ribbon Hopf algebra, it is unimodular.

Factorizability II

Under Majid's transmutation, we find the Hopf algebra structure of the *end* of H-mod, $\underline{H}$. It carries a *copairing* $\underline{w} \in \underline{H}$ given by $\underline{w} := S(M') \otimes M''$

Proposition (Beliakova and de Renzi (2023))

The category H-mod is factorizable iff $\underline{w}$ is non-degenerate, that is the Drinfeld map

$$\begin{aligned} D : \mathcal{E}^* &\to \mathcal{E} \\ e^* &\mapsto e^*(\underline{w}')\underline{w}'' \end{aligned}$$

is invertible.

It follows that if a unimodular ribbon H is factorizable then

$$\lambda(\underline{w}')\underline{w}'' = \underline{w}'\lambda(\underline{w}'') = \lambda(S(M'))M'' = S(M')\lambda(M'') = \Lambda,$$

So if $\lambda(S(M'))M'' \neq \Lambda$ then H-mod is non-factorizable.

Example

For $u_q\mathfrak{sl}_2$, at q an even root of unity this is $\lambda(S(M'))M'' = g\Lambda$, g an invertible element of $u_q\mathfrak{sl}_2$.

$$\mathcal{Z}_{(2)}(u_q\mathfrak{sl}_2) = \{\mathbb{1}, \mathcal{S}_{-1}\}$$

Strong non-factorizability

Definition (Faes-M'25)

A unimodular, quasitriangular Hopf algebra (H, R) with a (left) integral $\lambda \in H^*$ is called *strongly non-factorizable* if any of the two equivalent conditions is fulfilled

$$\lambda(S(M'))M'' = 0 \text{ or } S(M')\lambda(M'') = 0.$$

Any strongly non-factorizable Hopf algebra is non-factorizable.

Definition

Elements $g, h \in H$ have a *diagonal relation* if there exists $\gamma \in \mathbb{C}$ such that $gh = \gamma hg$.

Strategy

Construct such a unimodular ribbon Hopf algebra H that

- it is generated by grouplike and nilpotnent generators (usually pointed)
- there are nilpotent generators $X^{\pm}$ that do not appear in R
- generators $X^{\pm}$ do appear in Λ
- all new relations involving $X^{\pm}$ are diagonal.

Therefore,

$$\lambda(S(M'))M'' = S(M')\lambda(M'') = 0.$$

Biproducts with $u_q\mathfrak{sl}_2$ I

The Hopf algebra $u_q\mathfrak{sl}_2$ where q, is a primitive r-th root of unity, $r \equiv 0 \mod 8$, $r' = r/2$, $r'' = r/4$, is the algebra generated by elements K, E, and F, satisfying the relations

$$K^{r'} = \mathbf{1}, \;\; E^{r'} = F^{r'} = 0$$

$$KEK^{-1} = q^2 E, \;\; KFK^{-1} = q^{-2} F, \;\; [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The Hopf structure is given by

$$\epsilon(K) = 1, \;\; \epsilon(E) = \epsilon(F) = 0,$$

$$\Delta(K) = K \otimes K, \;\; \Delta(E) = \mathbf{1} \otimes E + E \otimes K, \;\; \Delta(F) = F \otimes \mathbf{1} + K^{-1} \otimes F$$

and

$$S(K) = K^{-1} = K^{r'-1}, \;\; S(E) = -EK^{-1}, \;\; S(F) = -KF.$$

Recall also the notation for quantum integers, for each $k \in \mathbb{Z}_{r'}$:

$$\{k\} := q^k - q^{-k}, \;\; [k] := \frac{\{k\}}{\{1\}}, \;\; [k]! := [k][k-1]\dots[1].$$

Biproducts with $u_q\mathfrak{sl}_2$ II

The Hopf algebra $u_q\mathfrak{sl}_2$ is

1. unimodular with a two-sided cointegral

$$\Lambda := \frac{\{1\}^{r'-1}}{\sqrt{r''}[r'-1]!}\sum_{a=0}^{r'-1} E^{r'-1}F^{r'-1}K^a,$$

   and a left integral

$$\lambda(E^bF^cK^a) := \begin{cases} \frac{\sqrt{r''}[r'-1]!}{\{1\}^{r'-1}} & \text{if } E^bF^cK^a = E^{r'}F^{r'}K^{r'} \\ 0 & \text{otherwise.} \end{cases}$$

2. quasitriangular with $R := D\Theta$, where

$$D\Theta := \frac{1}{r'}\sum_{b,c=0}^{r'-1} q^{-2bc}K^b \otimes K^c \sum_{a=0}^{r'-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2}} E^a \otimes F^a.$$

Biproducts with $u_q\mathfrak{sl}_2$ III

Definition Nenciu (2004)

Define $H(\mathbf{m}, t, \mathbf{d}, \mathbf{u})$ to be the Hopf algebra generated by grouplike generators $\mathbf{K} = (K_1, \ldots, K_s)$ (such that the corresponding group G is of even order), and skew-primitive generators $\mathbf{X} = (X_1, \ldots, X_t)$, and with the relations

$$K_a^{m_a} = \mathbf{1} \qquad K_a K_b = K_b K_a \qquad K_a X_k := \xi_a^{d_{ka}} X_k K_a$$
$$X_k^2 = 0 \qquad X_l X_k := \boldsymbol{\xi}^{\mathbf{d}_k \cdot \mathbf{u}_l} X_k X_l,$$

Hopf structure determined by

$$\epsilon(K_a) := 1 \qquad \Delta(K_a) := K_a \otimes K_a$$
$$\epsilon(X_k) := 0 \qquad \Delta(X_k) := \mathbf{1} \otimes X_k + X_k \otimes \mathbf{K}^{\mathbf{u}_k},$$
$$S(K_a) = K_a^{-1} := K_a^{m_a - 1} \qquad S(X_k) := -X_k \mathbf{K}^{-\mathbf{u}_k},$$

for $a, b = 1, \ldots, s$, $s := |\mathbf{m}|$ and $k, l = 1, \ldots, t$. Here $\mathbf{m} \in \mathbb{Z}^s_{>0}$, $\boldsymbol{\xi} := (\exp^{2\pi/m_1}, \ldots, \exp^{2\pi/m_s})$ and $\mathbf{d}$, $\mathbf{u} \in \mathrm{Mat}_{t\times s}(\mathbb{Z})$, satisfying certain conditions.

Biproducts with $u_q\mathfrak{sl}_2$ IV

Example

Let $u_q\mathfrak{sl}_2 \ltimes N_2$, be the Hopf algebra generated by K, E, F with the usual relations relations and morphisms, as well as K_1, K_2, K_3, $X^\pm, Z^\pm$ with the following relations

$$K_a^4 = \mathbf{1}, \quad K_a X^\pm = \pm i X^\pm K_a$$

$$K_1 Z^\pm K_1^{-1} = Z^\pm, \;\; K_2 Z^\pm K_2^{-1} = -Z^\pm, \;\; K_3 Z^\pm K_3^{-1} = \pm i Z^\pm$$

$$\{X^\pm, X^\pm\} = \{X^\pm, X^\mp\} = \{Z^\pm, X^\pm\} = \{Z^\pm, X^\mp\} = \{Z^\pm, Z^\pm\} = \{Z^\pm, Z^\mp\} = 0$$

With new relations

$$[K_a, K] = [K_a, E] = [K_a, F] = 0,$$

$$\{X^\pm, K\} = \{Z^\pm, K\} = \{X^\pm, E\} = \{Z^\pm, E\} = [X^\pm, F] = [Z^\pm, F] = 0.$$

Let also $\bar{L} := K^{r''} K_3^2$ as a shorthand, note that this time $\bar{L}^2 = \mathbf{1}$. The Hopf structure is defined by

$$\epsilon(K_a) = 1, \;\; \epsilon(X^\pm) = \epsilon(Z^\pm) = 0,$$

$$\Delta(K_a) = K_a \otimes K_a, \;\; \Delta(X^\pm) = \mathbf{1} \otimes X^\pm + X^\pm \otimes (K^{r''} K_1 K_2)^{\pm 1}, \;\; \Delta(Z^\pm) = \mathbf{1} \otimes Z^\pm + Z^\pm \otimes \bar{L}$$

$$S(K_a) = K_a^{-1} \;\; S(X^\pm) = -X^\pm (K^{r''} K_1 K_2)^{\mp 1} \;\; S(Z^\pm) = -Z^\pm \bar{L}.$$

at $r = 8$ the dimension is $4^4 \times 2^4 \times 2^2 \times 2^2 = 2^{16} = 65536$.

Biproducts with $u_q\mathfrak{sl}_2$ V

The algebra $u_q\mathfrak{sl}_2 \ltimes N_2$ is

1. unimodular with a two-sided cointegral

$$\Lambda := \frac{\{1\}^{r'-1}}{\sqrt{r''}[r'-1]!}\sum_{a=0}^{r'-1}\sum_{b,c,d=0}^{4} E^{r'-1}F^{r'-1}K^aK_1^bK_2^cK_3^dX^+X^-Z^+Z^-,$$

and a left integral expressed on the monomial basis by

$$\lambda_L(E^eF^fK^aK_1^bK_2^cK_4^d(X^+)^g(X^-)^h(Z^+)^i(Z^-)^j) := \\ = \begin{cases} \frac{\sqrt{r''}[r'-1]!}{\{1\}^{r'-1}} & \text{if } = E^{r'}F^{r'}K^{r'}X^+X^-Z^+Z^- \\ 0 & \text{otherwise.} \end{cases},$$

2. quasitriangular, with the R-matrix $R := R_{\mathbf{z}}\bar{R}_{\boldsymbol{\alpha}}D\Theta$, where D, Θ form the R-matrix of $u_q\mathfrak{sl}_2$ and

$$R_{\mathbf{z}} := \frac{1}{64}\sum_{\mathbf{v},\mathbf{w}\in\mathbb{Z}_4^3} i^{-\mathbf{v}\mathbf{w}^T}(K_1,K_2,K_3)^{\mathbf{w}} \otimes (K_1,K_2,K_3)^{\mathbf{vz}},$$

$$\bar{R}_\alpha := \exp\left(\alpha(Z^+\otimes \bar{L}Z^- - Z^-\otimes \bar{L}Z^+)\right),$$

where $\mathbf{z} = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ and $\alpha \in \mathbb{C}$,

Representation theory of Nenciu biproducts I

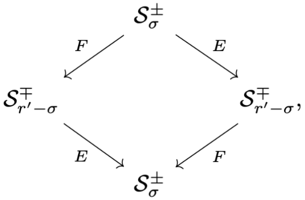
By the results of Gainutdinov, Semikhatov, Tipunin, and Feigin (2006) the full system of idempotents for $u_q\mathfrak{sl}_2$ has the form

$$I_{p,\sigma} = \varphi_p^K \mathbf{e}_\sigma, \quad p \in \mathbb{Z}_{r'}, \quad 1 \leq \sigma \leq r' - 1, \quad p - \sigma \equiv 1 \mod 2, \tag{1}$$

where $\mathbf{e}_s$ are central idempotents and

$$\varphi_p^K = \frac{1}{r'} \sum_{j=0}^{r'-1} q^{-2pj} K^j,$$

is the projector to the pth eigenspace of K. It was shown in that the corresponding indecomposable projective modules, $\mathcal{P}_\sigma^+ := u_q\mathfrak{sl}_2 I_{\sigma-1,\sigma}$ and $\mathcal{P}_{r'-\sigma}^- := u_q\mathfrak{sl}_2 I_{-\sigma-1,\sigma}$, have the shape



Representation theory of Nenciu biproducts II

Definition

Let for a grouplike generator $K_a \in H(\mathbf{m}, t, \mathbf{d}, \mathbf{u})$, and the corresponding root of unity ξ_a

$$\varphi_p^a := \frac{1}{m_a} \sum_{j=0}^{m_a} \xi_a^{-jp} K_a^j.$$

Definition

Let for any $\mathbf{p} \in \mathbb{Z}_{\mathbf{m}}$,

$$\omega_{\mathbf{p}} := \prod_{a=1}^{s} \varphi_{p_a}^a,$$

Proposition

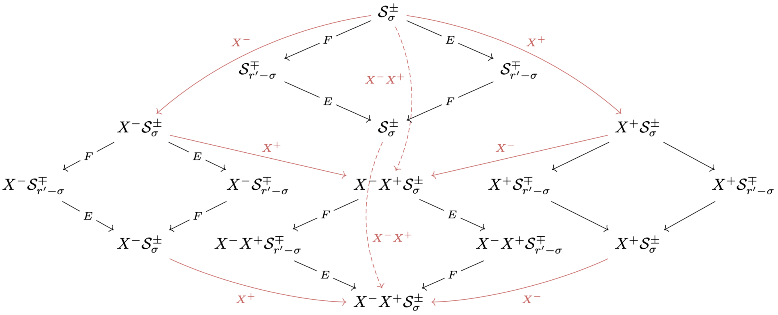
For all $\mathbf{p} \in \mathbb{Z}_{\mathbf{m}}$, $\omega_{\mathbf{p}}$ form a full system of idempotents in $H(\mathbf{m}, t, \mathbf{d}, \mathbf{u})$.

Representation theory of Nenciu biproducts III

Proposition

Let $U \ltimes H$ with $U = u_q\mathfrak{sl}_2$ and let $H = H(\mathbf{m}, t, \mathbf{d}, \mathbf{u})$ have the full system of idempotents $\{\omega_{\mathbf{p}}\}_{\mathbf{p}\in\mathbb{Z}_{\mathbf{m}}}$. Then the full system of idempotents of $U \ltimes H$ is

$$\Omega_{p,\sigma,\mathbf{p}} := I_{p,\sigma}\omega_{\mathbf{p}}, \quad \text{for } \ p = 0, \dots, r' - 1, \ \ \mathbf{p} \in \mathbb{Z}_{\mathbf{m}}, \ \ p - s \equiv 1 \mod 2.$$



The relations between K, E and $X_k^\pm, Z_l^\pm$ introduce a $\mathbb{Z}_2$-grading with respect to K-eigenvalues

Representation theory of Nenciu biproducts IV

Proposition

The Hopf algebra $u_q\mathfrak{sl}_2 \ltimes \mathrm{N}_2$ admits only strongly non-factorizable ribbon structures and no triangular ones. The symmetric centre $\mathcal{Z}_{(2)}(u_q\mathfrak{sl}_2 \ltimes \mathrm{N}_2\text{-mod})$ is non-semisimple. Moreover, it containts infinitely many indecomposable objects.

Let $\Xi_{k,\varepsilon,\delta}$ for $k \in \mathbb{Z}_{\geq 0}$ and $\varepsilon, \delta \in \{0,1\}$, be the vector space $\mathbb{C}^{2k+\varepsilon+\delta+1}$ equipped with

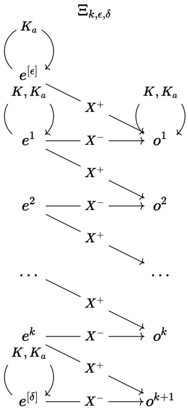
$$X^+ e^\gamma = o^{\gamma+1}, \quad X^- e^\gamma = o^\gamma \quad X^\pm o^\beta = 0,$$
$$X^+ e^{[\varepsilon]} = o^1, \quad X^- e^{[\varepsilon]} = 0, \quad X^+ e^{[\delta]} = o^1, \quad X^- e^{[\delta]} = o^{k+1}.$$

for $\gamma = 1, \ldots, k$

$$K_a e^\alpha = i^{v_a+2\alpha} e^\alpha, \quad \text{for } \alpha = 1, \ldots, k, \varepsilon, \delta,$$
$$K_a o^\beta = i^{v_a+2\beta-1} o^\beta, \quad \text{for } \beta = 1, \ldots, k+1,$$

and for all α, β

$$Ke^\alpha = -e^\alpha, \quad e^\alpha = Fe^\alpha = Z^\pm e^\alpha = 0$$
$$Ko^\beta = o^\beta, Eo^\beta = Fo^\beta = Z^\pm o^\beta = 0.$$



Bibliography I

Anna Beliakova and Marco de Renzi. Kerler–lyubashenko functors on 4-dimensional 2-handlebodies. *International mathematics research notices*, Feb 2023. doi: https://doi.org/10.1093/imrn/rnac039. URL https://arxiv.org/abs/2105.02789.

Ivelina Bobtcheva and Riccardo Piergallini. A universal invariant of four-dimensional 2-handlebodies and three-manifolds, 2006. URL https://arxiv.org/abs/math/0612806.

Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. *Journal of Topology*, 7(4):1005–1053, 2014.

Francesco Costantino, Nathan Geer, Benjamin Haïoun, and Bertrand Patureau-Mirand. Skein (3+1)-tqfts from non-semisimple ribbon categories, 2023. URL https://arxiv.org/abs/2306.03225.

A. M. Gainutdinov, A. M. Semikhatov, I. Yu. Tipunin, and B. L. Feigin. Kazhdan-lusztig correspondence for the representation category of the triplet w-algebra in logarithmic cft. *Theoretical and Mathematical Physics*, 148(3):1210–1235, Sep 2006. doi: https://doi.org/10.1007/s11232-006-0113-6.

Thomas Kerler and Volodymyr V Lyubashenko. *Non-semisimple topological quantum field theories for 3-manifolds with corners*, volume 1765. Springer Science & Business Media, 2001.

Bibliography II

Adriana Nenciu. Quasitriangular pointed hopf algebras constructed by ore extensions. *Algebras and Representation Theory*, 7(2):159–172, May 2004. doi: https://doi.org/10.1023/b:alge.0000026785.03997.60.

Nicolai Reshetikhin and Vladimir G Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inventiones mathematicae*, 103(1):547–597, 1991.

Kenichi Shimizu. Non-degeneracy conditions for braided finite tensor categories, 2016. URL https://arxiv.org/abs/1602.06534.