Coproduct of T-series for Yangians in type A

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Yangians

Drinfeld presentation

The Yangian Y of a finite-dimensional complex Lie algebra $\mathfrak{g}$ is a deformation of the enveloping algebra of the current algebra $\mathfrak{g}[u]$. It can be defined by generators $x_{i,n}^{\pm}, \xi_{i,n}$ and relations. The $\xi_{i,n}$ generate a commutative subalgebra, called Drinfeld-Cartan subalgebra.

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One of the main open question for this presentation is the coproduct of these generators. For instance, in the case $\mathfrak{sl}_2$, Molev gave the following formula for the generating series $\xi(z)$:

$$\Delta(\xi(z)) = (1 \otimes \xi(z)) \left(\sum_{k=0}^{\infty} (-1)^k (k+1) (x^-(z+1))^k \otimes (x^+(z+1))^k \right) (\xi(z) \otimes 1) .$$

T-series

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Remark

Evaluations of T-series at certain representations of Yangians were studied by Gautam-Wendlandt (2021) and Hernandez-Zhang (2021).

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Commuting relations for T-series

The series $T_i(z)$ commute with $x_j^{\pm}(z)$ for $j \neq i$ and we have the following relations for $1 \leqslant i \leqslant n$:

$$T_i(z) x_{i,n}^- T_i(z)^{-1} = x_{i,n+1}^- - z x_{i,n}^-$$
$$T_i(z)^{-1} x_{i,n}^+ T_i(z) = x_{i,n+1}^+ - z x_{i,n}^+.$$

Coproduct of T-series

The coproduct of the T-series can be factorized as follow :

$$\Delta(T_i(z)) = (1 \otimes T_i(z))\Theta_i(z)(T_i(z) \otimes 1)$$

where $\Theta_i(z)$ can be interpreted as an associator for some representations of shifted Yangians. Zhang (2024) proved $\Theta_i(z)$ is locally polynomial in z.

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Theorem (M.)

Let $\mathfrak{g}$ a Lie algebra of type A_n. Denote by $(E_{jk})_{1\leqslant j,k\leqslant n+1}$ the elementary matrices in $\mathfrak{sl}_{n+1}$. Then, for $1 \leqslant i \leqslant n$:

$$\Theta_i(z) = \exp\left(\sum_{1\leqslant j\leqslant i<k\leqslant n+1} E_{kj} \otimes E_{jk}\right).$$

Computation of the coproduct

Intertwining equations

The intertwining equations of $\Theta_i(z)$ encode the fact the series have to commute with the actions of the generators of Y. In fact, it suffices to satisfy the commutation with the generators $x^+_{j,0}$ to determine Θ_i. These equations are verified for all type of Lie algebra, but can be complicated to write.

In type A

Intertwining relations

In type A_n, we get a system of n equations for $\Theta_i(z)$.

For $j \neq i$: $\left[x^+_{j,0} \otimes 1 + 1 \otimes x^+_{j,0}, \Theta_i(z)\right] = 0.$

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The last equation is :

$$\left[x^+_{i,0} \otimes 1 + 1 \otimes x^+_{i,1} - z(1 \otimes x^+_{i,0}), \Theta_i(z)\right] = \Theta_i(z) \left(z h_{i,0} \otimes x^+_{i,0} - \sum_{k=i+2}^{n+1} E_{k,i+1} \otimes E_{i,k} + \sum_{j=1}^{i-1} E_{i,j} \otimes E_{j,i+1} \right).$$

Applications

- Construction of R-matrices for standard Yangians (following Zhang 2024).

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- Explicit computation for certain quotients of Yangians, and consequences for their representation theories (in case $\mathfrak{sl}_2$ for now (M. 2025)).

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Shifted case

In fact, all these constructions in the original article are considered for shifted Yangians. Yet, our formula can be generalized by classical zigzag arguments.

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