Late-time waveforms and Dualities

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work in progress
with
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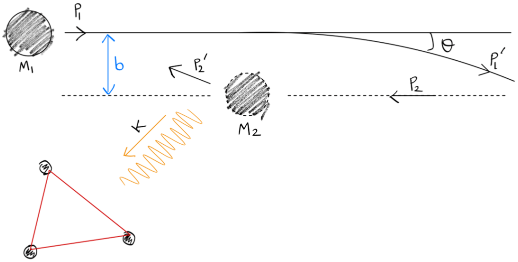
Gravitational wave physics requires **high precision** (LIGO/LISA etc).



- EOM: $\partial^2 h_{\mu\nu} = T_{\mu\nu}$.
- Complicated: spin, tidal effects, radiation, modified theories...
- Simpler: $\partial^\mu F_{\mu\nu} = j_\nu$

Can also consider **dual solutions** to these theories:

For two-forms $X_{\mu\nu}$, the Hodge dual is $\tilde{X}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}X^{\rho\sigma}$.

Maxwell

$$\partial^\mu F_{\mu\nu} = 0, \quad \partial^\mu \tilde{F}_{\mu\nu} = 0.$$

If $F_{\mu\nu}$ solves these, so does

$$F'_{\mu\nu} = \cos\theta\, F_{\mu\nu} + \sin\theta\, \tilde{F}_{\mu\nu}.$$

Charge $e = Q\cos\theta$, $g = Q\sin\theta$

Linearised GR

$$\partial^\mu C_{\mu\nu\rho\sigma} = 0, \quad \partial^\mu \tilde{C}_{\mu\nu\rho\sigma} = 0.$$

If C solves these, so does

$$C'_{\mu\nu\rho\sigma} = \cos\theta\, C_{\mu\nu\rho\sigma} + \sin\theta\, \tilde{C}_{\mu\nu\rho\sigma}.$$

Mass $m = M\cos\theta$, $n = M\sin\theta$

Electric-Magnetic Duality

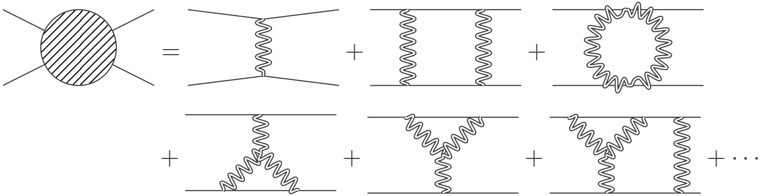
One-parameter family of solutions to Maxwell and linearised GR, e.g.
Coulomb $(Q, 0)$ $\longleftrightarrow$ Dyon (e, g)
Schwarzschild $(M, 0)$ $\longleftrightarrow$ Taub-NUT (m, n)

This talk: Mostly about classical dyons, explored with **scattering amplitudes**.

Many benefits

- on-shell data
- factorisation
- recursion
- spinor-helicity
- soft theorems
- unitarity cuts

Some downsides: **quantum baggage**, Fourier transforms, weak field...



Classical physics: $\hbar \to 0$ – use KMOC.

Classical observables from amplitudes

$$\Delta\mathcal{O} = \lim_{\hbar\to 0}\left[\langle\psi|\mathbb{S}^\dagger\mathbb{O}\mathbb{S}|\psi\rangle - \langle\psi|\mathbb{O}|\psi\rangle\right]$$

Start from a two-particle quantum state

$$|\psi\rangle = \int_{p_1,p_2} \varphi(p_1,p_2)e^{i\sum b_i\cdot p_i}\,|p_1,p_2\rangle$$

Expand $\mathbb{S} = 1 + i\mathbb{T}$ and consider operators $\mathbb{O} = \{\mathbb{P}^\mu, \mathbb{J}^{\mu\nu}, \mathbb{W}^\mu, \mathbb{A}^\mu, \mathbb{h}^{\mu\nu}, ...\}$

Observables:

The impulse Δp^μ, the angular-impulse $\Delta J^{\mu\nu}$, waveform $\Delta h_{\mu\nu}$ and others.

Observables directly from amplitudes: **No EOM required**.

Let's see how to compute these.

The **impulse** is given by

$$\Delta p^\mu = \langle \psi | \mathbb{S}^\dagger [\mathbb{P}^\mu, \mathbb{S}] | \psi \rangle \simeq \langle \psi | i [\mathbb{P}^\mu, \mathbb{T}] | \psi \rangle + \mathcal{O}(TT^\dagger)$$

Amplitudes:

$$\langle p_1', p_2' | \mathbb{T} | p_1, p_2 \rangle = A_4[p_1, p_2 \to p_1', p_2'] = \hat{\delta}^{(4)}(\sum p_i) \mathcal{A}_4[p_1, p_2 \to p_1', p_2']$$

Leading order

$$\Delta p^\mu = i \int \hat{d}^4 q \hat{\delta}(2p_1 \cdot q) \hat{\delta}(2p_2 \cdot q) e^{-iq\cdot b} q^\mu \mathcal{A}_4[p_1, p_2 \to p_1 + q, p_2 - q]$$
$$= i \int_q \hat{\delta}_1 \hat{\delta}_2 e^{-iq\cdot b} q^\mu \mathcal{A}_4$$

Equivalent to the leading-order result in e.g. scalar QED

$$\Delta p^\mu = \int d\tau \frac{dp^\mu}{d\tau} = \int d\tau \; eF^{\mu\nu}(x(\tau)) u_\nu(\tau)$$

The **angular impulse** is given by

$$\Delta J^{\mu\nu} = \langle\psi|\mathbb{S}^\dagger[\mathbb{J}^{\mu\nu},\mathbb{S}]|\psi\rangle \simeq \langle\psi|i[\mathbb{X}^{[\mu}\mathbb{P}^{\nu]},\mathbb{T}]|\psi\rangle + \mathcal{O}(TT^\dagger)$$

expand in amplitude, use $\mathbb{X}^\mu = i\partial_p^\mu$,

$$\Delta J_i^{\mu\nu} = \int_{q_1,q_2} \hat{\delta}_1\hat{\delta}_2 e^{-ib\cdot q_1}\left((p_i+q_i)^{[\mu}\partial^{\nu]}_{p_i+q_i} + p_i^{[\mu}\partial^{\nu]}_{p_i}\right)\hat{\delta}^{(4)}(q_1+q_2)\mathcal{A}_4$$

... tedious algebra, IBPs, defining $y_i^\mu = p_i^\mu - \frac{p_i\cdot p_j}{m_j^2}p_j^\mu$

$$\Delta J_i^{\mu\nu} = b^{[\mu}\Delta p_i^{\nu]} - \int_q \hat{\delta}_1\hat{\delta}_2 e^{-ib\cdot q}\left[\frac{y_i^{[\mu}p_i^{\nu]}}{y_i\cdot p_i} + p_i^{[\mu}\partial^{\nu]}_{p_i}\right]\mathcal{A}_4$$

Second term important, see later!

The **waveform** is given by

$$\Delta F^{\mu\nu} = \langle\psi|\mathbb{S}^\dagger[\mathbb{F}^{\mu\nu},\mathbb{S}]|\psi\rangle \simeq \langle\psi|i[\mathbb{F}^{\mu\nu},\mathbb{T}]|\psi\rangle + \mathcal{O}(TT^\dagger)$$

As an operator, $\mathbb{F}^{\mu\nu} \sim a^\dagger + a$, we get

$$\begin{aligned}\Delta F^{\mu\nu}(k) &= \sum_\eta \int_{q_1,q_2} e^{-ib\cdot q_1} k^{[\mu}\bar{\epsilon}_\eta^{\nu]} \langle p_1', p_2'|a^\eta(k)\mathbb{T}|p_1,p_2\rangle \\ &= \sum_\eta \int_{q_1,q_2} e^{-ib\cdot q_1} k^{[\mu}\bar{\epsilon}_\eta^{\nu]} A_5[p_1,p_2 \to p_1+q_1, p_2+q_2, k^\eta]\end{aligned}$$

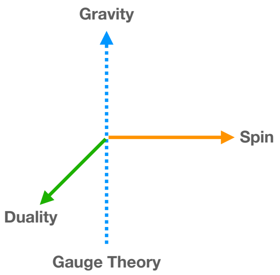
In gravity: $k^{[\mu}\bar{\epsilon}_\eta^{\nu]} \to k^{[\mu}\bar{\epsilon}_\eta^{\nu]}k^{[\rho}\bar{\epsilon}_\eta^{\sigma]}$

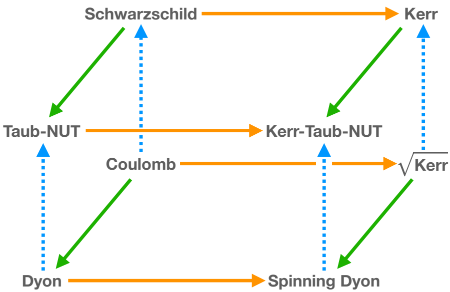
$$\Delta R^{\mu\nu\rho\sigma}(k) = \sum_\eta \int_{q_1,q_2} e^{-ib\cdot q_1} k^{[\mu}\bar{\epsilon}_\eta^{\nu]}k^{[\rho}\bar{\epsilon}_\eta^{\sigma]} M_5[p_1,p_2 \to p_1+q_1, p_2+q_2, k^\eta]$$

Amplitude Solution Webs

Observables $\leftrightarrow$ Amplitudes $\leftrightarrow$ Classical solutions: 4pt amplitudes generate impulses, 5pt amplitudes generate waveforms.

Deform amplitudes $\mapsto$ generate new classical solutions!

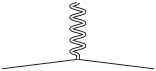




We can use on-shell duality rotations to scatter Dyons or Taub-NUTs.

On-Shell Electric-Magnetic Duality

We can apply duality rotations **on-shell** to 3pt amplitudes:

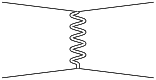


$$= Q_1^s M_1 (u_1 \cdot \epsilon^h(q))^s \quad \mapsto \quad Q_1^s M_1 (u_1 \cdot \epsilon^h(q))^s e^{ih\theta_1},$$

where $Q_1^1 = \sqrt{2} Q$, $Q_1^2 = \frac{1}{\sqrt{2}} \kappa M_1$.

We identify $Qe^{i\theta} = e + ig$ or $Me^{i\theta} = m + in$.

We can construct four-particle amplitudes by factorisation



$$= \sum_{h=\pm} \frac{Q_1^s M_1 Q_2^s M_2}{q^2} (u_1 \cdot \epsilon^h(q) u_2 \cdot \epsilon^{-h}(q))^s e^{ih(\theta_1 - \theta_2)}$$

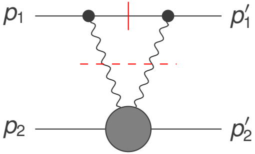
$$= \frac{f(p_i, \theta_i)}{q^2} + \frac{g(p_i, \theta_i)}{q^2} \frac{\varepsilon(p_1, p_2, n, q)}{n \cdot q}$$

where we used

$$\epsilon_\mu^\pm \epsilon_\nu^\mp = -\frac{1}{2} \left(\eta_{\mu\nu} - \frac{n_{(\mu} q_{\nu)}}{n \cdot q} \right) \pm i \frac{\varepsilon_{\mu\nu}(n, q)}{2 n \cdot q}.$$

This is general (gauge + gravity). Let's look at one loop for dyons only

We can also look at one-loop amplitudes with classical parts

$\mathcal{A}_4 =$ 

$+ \quad 1 \longleftrightarrow 2$

On the cuts, amplitude becomes

$$A_4^{(1)} = i\pi \int_\ell \sum_{h_1, h_2 = \pm} \frac{A_3[p_1, \ell]^{h_1} A_3[p_1', q - \ell]^{h_2} A_4[p_2, p_2', -\ell, -q + \ell]^{-h_1, -h_2}}{\ell^2 (q - \ell)^2}$$

Surprisingly, at one-loop, $\mathcal{A}_4^{Dyons} = \mathcal{A}_4^{Elec}$. All phases cancel!

Easily understood: only opposite helicity exchange at one-loop.

Amplitude governed by $e_1 e_2 + g_1 g_2$ only, matches 2PL worldline calculation.

Gravity less easy, as we will discuss!

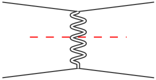
Dirac-Misner Strings

Problem: Amplitudes apparently depend on n^μ...

Famous **Dirac string** in EM or **Misner line** in gravity.

Various interpretations, but from amplitudes: $n \cdot \epsilon^{\pm}(q) = 0$.

Is there *really* n-dependence? Classical part is given by



Propagators is **cut**: $q^2 = 0 = q \cdot p_i$.

$$n^\mu = n'^\mu + c_1 q^\mu + c_2 p_1^\mu + c_3 p_2^\mu$$

on this support, we find

$$\frac{\varepsilon(p_1, p_2, n, q)}{n \cdot q} = \frac{\varepsilon(p_1, p_2, n', q)}{n' \cdot q}$$

No real n-dependence classically, just gauge.

Plug in general dyonic-amplitude

$$\Delta p^\mu_{Dyons} = \int_{q_\perp} e^{-ib\cdot q}\left(\frac{f(p_i,\theta_i)}{q^2}q^\mu + \frac{g(p_i,\theta_i)}{q^2}\varepsilon^\mu(p_1,p_2,q)\right)$$
$$= -\frac{1}{4\pi M_1 M_2\sqrt{\gamma^2-1}}\left(\frac{f(p_i,\theta_i)}{b^2}b^\mu + \frac{g(p_i,\theta_i)}{b^2}\varepsilon^\mu(p_1,p_2,b)\right)$$

Using $q^{[\mu}\varepsilon^{\nu\rho\sigma\tau]} = 0$ and $q\cdot p_i = 0$. Manifestly n-independent!

Matches classical expression, in E&M

$$\Delta p^\mu = \int d\tau\left(eF^{\mu\nu}(x(\tau)) + g\tilde{F}^{\mu\nu}(x(\tau))\right)u_\nu(\tau)$$

or in gravity

$$\Delta p^\mu = -\int d\tau\,(M\omega^{\mu\nu\rho}(x(\tau)) + N\tilde{\omega}^{\mu\nu\rho}(x(\tau)))\,u_\nu(\tau)u_\rho(\tau)$$

No need for one-loop, adds nothing new.

Plug in explicit four-dyon amplitude: two charge invariants

$$\mathcal{A}_4 = \frac{4(e_1 e_2 + g_1 g_2)(p_1 \cdot p_2)}{q^2} + \frac{(e_1 g_2 - e_2 g_1)}{q^2} \frac{\varepsilon(p_1, p_2, n, q)}{n \cdot q}$$

to find

$$\Delta J^{\mu\nu}_{dyon} = b^{[\mu} \Delta p^{\nu]}_{dyon} - \frac{(e_1 e_2 + g_1 g_2) u_1^{[\mu} u_2^{\nu]}}{4\pi(\gamma^2 - 1)^{3/2}} \ln b/b_0$$

Notice that if $\Delta p^\mu \sim b^\mu$, then $b^{[\mu} \Delta p^{\nu]} = 0$: dyon generates angular momentum!

$$b^{[\mu} \Delta p^{\nu]}_{dyon} = \frac{e_1 g_2 - e_2 g_1}{2\pi \sqrt{\gamma^2 - 1}} \varepsilon^{\mu\nu}(u_1, u_2)$$

Second term more interesting: no mixed $e_1 g_2 - e_2 g_1$ contribution! Similar term in gravity: $(M_1 N_2 - N_1 M_2)$ also absent.

No Taub-NUT contribution to the late-time log terms. Let's calculate it classically.

Dyonic Lorentz force

$$m_a \frac{d^2 x_a^\mu(\tau)}{d\tau^2} = e_a F_b^{\mu\nu}(x_a(\tau)) \frac{dx_\nu(\tau)}{d\tau} + g_a \tilde{F}_b^{\mu\nu}(x_a(\tau)) \frac{dx_\nu(\tau)}{d\tau},$$

Straight line plus a recoil term: $x_a^\mu(\tau) = b^\mu + u_a^\mu \tau + z_a^\mu(\tau)$. Asymptotic fields

$$F_b^{\mu\nu} = \frac{e_b}{4\pi} \frac{u_b^{[\mu} u_a^{\nu]}}{\tau^2 \left(\gamma^2 - 1\right)^{3/2}} - \frac{g_b}{4\pi} \frac{\varepsilon^{\mu\nu\rho\sigma} u_{b\rho} u_{a\sigma}}{\tau^2 \left(\gamma^2 - 1\right)^{3/2}} + \mathcal{O}\left(\tau^{-3}\right)$$

Plug in to find **no** $e_1 g_2 - e_2 g_1$ **term**

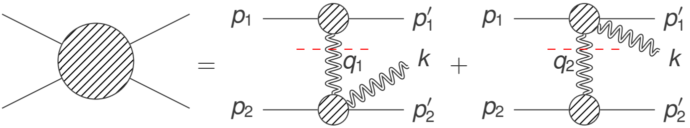
$$m_a \frac{d^2 z_a^\mu(\tau)}{d\tau^2} = \sum_{b \neq a} \frac{e_a e_b + g_a g_b}{4\pi} \frac{u_b^\mu + \gamma u_a^\mu}{\tau^2 \left(\gamma^2 - 1\right)^{3/2}} + \mathcal{O}\left(\tau^{-3}\right)$$

$J^{\mu\nu}$ from late-time path $x^\mu = b^\mu + u^\mu \tau + c_a^\mu \ln \tau/\tau_0$ to find

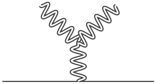
$$\Delta J_a^{\mu\nu} = x^{[\mu} \Delta p_a^{\nu]} + \Delta x^{[\mu} p_a^{\nu]} = b^{[\mu} \Delta p_a^{\nu]} - \frac{(e_a e_b + g_a g_b) u_a^{[\mu} u_b^{\nu]}}{4\pi(\gamma^2 - 1)^{3/2}} \ln \tau/\tau_0$$

Classical ‘late-time’ log emerges as a large-b log in KMOC.

To compute the Waveform, we need the 5pt amplitude



Problem: fully non-linear GR lacks global helicity rotation symmetry.

How do we dualise this? 

Similar in spirit to the Kerr Compton problem (no consistent Compton for higher spins > 2).

Won't solve this now – let's look at the soft limit instead.

Single soft-limit just requires 3pt amplitudes, which we know how to dualise

$$A_5[p_1, p_2 \to p_1', p_2', k \to 0] = \left(\sum_i S_i(k)\right) A_4[p_1, p_2 \to p_1', p_2']$$

First two orders **universal**

$$S^{\eta}_{-1} = \sum_{i=1,2} Q_i^s e^{i\eta\theta_i} \left(\frac{(p_i' \cdot \epsilon^{\eta}(k))^s}{p_i' \cdot k} - \frac{(p_i \cdot \epsilon^{\eta}(k))^s}{p_i \cdot k} \right),$$

$$S^{\eta}_{0} = \sum_{i=1,2} Q_i^s e^{i\eta\theta_i} \left(\frac{(p_i' \cdot \epsilon^{\eta}(k))^{s-1} \epsilon^{\eta}_{\mu} k_{\nu} J_i'^{\mu\nu}}{p_i' \cdot k} - \frac{(p_i \cdot \epsilon^{\eta}(k))^{s-1} \epsilon^{\eta}_{\mu} k_{\nu} J_i^{\mu\nu}}{p_i \cdot k} \right)$$

These are duality-safe in gauge and gravity.

These act on **amplitudes**, however: we want **classical** radiation. Let's see how to get it.

We want **classical** soft radiation, so we take the classical limit using KMOC.

Prescription:

- Take KMOC expectation of an operator $\langle \mathbb{O} \rangle \sim \langle a + a^\dagger \rangle$
- KMOC wavepackets localise classical particles on-shell, $p_i' = m_i u_i + q_i$ with $q_i \sim \hbar$
- Radiation momentum $k^\mu \sim \hbar$: expand in $q_i, k \ll p_i$
- Carefully take $\hbar \to 0$ limit, keeping $q_i/\hbar, k/\hbar$ as fixed classical wavenumbers
- Under the integral, expand 5pt into soft $\times$ 4pt
- Identify classical observables that emerge, e.g. impulse $\Delta p_i^\mu = \int_q q_i^\mu \mathcal{A}_4$, $\Delta J^{\mu\nu}$.

For $s = 1$, the leading order waveform is

$$\Delta F^{\mu\nu}_{-1}(k) = \sum_{\eta} \int_{q_1,q_2} k^{[\mu}\bar{\epsilon}^{\nu]}_{\eta} S^{\eta}_{-1}(k) A_4 = \int_{q_1,q_2} S^{\mu\nu}_{-1}(k) A_4 .$$

Where $S^{\mu\nu}_{-1}$ expanded around $p' = p + q \gg p + k$ is

$$S^{\mu\nu}_{-1} = \sum_i \left(e_i \frac{k^{[\mu} q_i^{\nu]}}{k \cdot p_i} - e_i \frac{(k \cdot q_i) k^{[\mu} p_i^{\nu]}}{(k \cdot p_i)^2} - ig \frac{k^{[\mu} \varepsilon^{\nu]}(k, p_i, q_i)}{k \cdot p_i} \right) + \cdots$$

Integrating against q_i and identifying the impulse $\Delta p_i^{\mu} = \int_q q^{\mu} \mathcal{A}_4$, we find

$$\Delta F^{\mu\nu}_{-1}(k) = \sum_i \left(e_i \frac{k^{[\mu} \Delta p_i^{\nu]}}{k \cdot p_i} - e_i \frac{(k \cdot \Delta p_i) k^{[\mu} p_i^{\nu]}}{(k \cdot p_i)^2} - ig \frac{k^{[\mu} \varepsilon^{\nu]}(k, p_i, \Delta p_i)}{(k \cdot p_i)^2} \right) .$$

For $s = 1$, the subleading order waveform is given by

$$\Delta F_0^{\mu\nu}(k) = \int_{q_1, q_2} \hat{\delta}_1 \hat{\delta}_2 S_0^{\mu\nu}(k) A_4$$

where, defining $\bar{J}^\mu = k_\rho J^{\mu\rho} = p^\mu k \cdot \partial_p - k \cdot p \partial_p^\mu$, is

$$S_0^{\mu\nu}(k) = \sum_i \left(e_i \frac{k^{[\mu} \bar{J}_i'^{\nu]}}{k \cdot p_i'} - e_i \frac{k^{[\mu} \bar{J}_i^{\nu]}}{k \cdot p_i} + ig \frac{\varepsilon^{\mu\nu}(k, \bar{J}_i')}{k \cdot p_i'} - ig \frac{\varepsilon^{\mu\nu}(k, \bar{J}_i)}{k \cdot p_i} \right)$$

expanding around $p' = p + q \gg p + k$ as before and identifying $\Delta J^{\mu\nu}$ and Δp^μ as before, we find

$$\Delta F_0^{\mu\nu}(k) = \sum_{i=1,2} e_i \left(\frac{k^{[\mu} \Delta \bar{J}^{\nu]}}{k \cdot p_i} - \frac{(k \cdot \Delta p_i) k^{[\mu} \bar{J}^{\nu]}}{(k \cdot p_i)^2} \right)$$
$$- \sum_{i=1,2} ig \left(\frac{k \cdot \Delta p_i \varepsilon^{\mu\nu}(\bar{J}_i, k) - k \cdot p_i \varepsilon^{\mu\nu}(\Delta \bar{J}_i, k)}{(k \cdot p_i)^2} \right)$$

For $s = 2$, we use the radiative Weyl. The leading waveform is

$$\Delta C^{\mu\nu\rho\sigma}_{-2} = \sum_\eta \int_{q_1,q_2} \left[k^{[\mu}\bar{\varepsilon}^{\nu]}_{\eta\ \alpha}(k)\right]\left[k^{[\rho}\bar{\varepsilon}^{\sigma]\alpha}_{\eta}(k)\right] S^{\eta}_{-2}(k)\, A_4 \ = \int_{q_1,q_2} S^{\mu\nu\rho\sigma}_{-2}(k)\, A_4$$

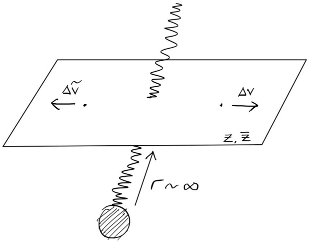
$$= \frac{\kappa}{2}\sum_i \Bigg[\cos(\theta_i)\left(\frac{k^{[\mu}\Delta p_i^{\nu]}}{k\cdot p_i} - \frac{(k\cdot\Delta p_i)\,k^{[\mu}p_i^{\nu]}}{(k\cdot p_i)^2}\right)k^{[\rho}p_i^{\sigma]} + \ (\mu\nu)\leftrightarrow(\rho\sigma)$$

$$- \ i\sin(\theta_i)\,\frac{k^{[\mu}\varepsilon^{\nu]}(k,p_i,\Delta p_i)}{(k\cdot p_i)^2}\,k^{[\rho}p_i^{\sigma]} \ + \ (\mu\nu)\leftrightarrow(\rho\sigma)\Bigg] \ + \cdots$$

where we played the same game as before, identifying the impulse Δp_i and dual impulse $\varepsilon(k, p_i, \Delta p_i)$ via their KMOC integrals.

Subleading waveform is work in progress...

Permanent change in the field at **null infinity** after a burst of radiation.



In retarded coordinates $(u, r, z, \bar{z})$, define

$$\mathcal{E}_A = \int du\, F_{\mu\nu} \bar{\ell}^\mu e^\nu_A,$$

This is the memory vector.

$\bar{\ell}^\mu$ defined such that $\bar{\ell} \cdot k = \omega$ and $e^\nu_A = \frac{\partial \hat{x}^\mu(z,\bar{z})}{\partial x^A}$ is purely angular.

After the wave, test electric charges $\tilde{e}$ experience an impulse (kick) on the sphere

$$\Delta v_A = \frac{\tilde{e}}{m} \mathcal{E}_A$$

In gravity, we get a permenant **displacement** Δx of test masses on the celestial sphere, no velocity kick. Just look at dyons today!

Dyonic Memory Effect

Using $k^\mu = \omega \ell^\mu$ and $e_A \cdot \ell = 0$, we get

$$\begin{aligned}\mathcal{E}_A &= \Delta F^{\mu\nu}_{-1} \bar{\ell}_\mu e_{A\nu} \\ &= \sum_i \left(e_i \frac{\Delta p_{iA}}{\ell \cdot p_i} - e_i \frac{(\ell \cdot \Delta p_i) p_{iA}}{(\ell \cdot p_i)^2} - ig \frac{\varepsilon(e_A, \ell, p_i, \Delta p_i)}{(\ell \cdot p_i)^2} \right) \\ &= \sum_i (e_i D_A + g_i \varepsilon_{AB} D^B) \frac{\ell \cdot \Delta p_i}{\ell \cdot p_i}\end{aligned}$$

where $D_A = e^\mu_A \frac{\partial}{\partial \ell^\mu}$ and we used

$$\varepsilon(e_A, \ell, p_i, \Delta p_i) = i \varepsilon_{AB} \Big((\Delta p \cdot \ell) p^B - (p \cdot \ell) \Delta p^B \Big)$$

Even nicer duality symmetric form

$$\mathcal{E}_A = \mathrm{Re} \sum_i Q_i e^{i\theta_i} (D_A + i \varepsilon_{AB} D^B) \frac{(\ell \cdot \Delta p_i)}{\ell \cdot p_i}$$

Dyonic test particles on the celestial sphere experience a **velocity kick**.
Direction determined by $e_i, g_i, \theta = [0, \pi/2]$.

We can now compute the subleading memory effect

$$\mathcal{E}_A = \Delta F^0_{\mu\nu} K^\mu e^\nu_A$$
$$= \sum_i \left[e_i \left(\frac{\ell_\mu \Delta J^\mu_{iA}}{\ell \cdot p_i} - \frac{(\ell \cdot \Delta p_i)\, \ell_\mu J^\mu_{iA}}{(\ell \cdot p_i)^2} \right) + g_i\, \epsilon_A{}^B \left(\frac{\ell_\mu \Delta J^\mu_{iB}}{\ell \cdot p_i} - \frac{(\ell \cdot \Delta p_i)\, \ell_\mu J^\mu_{iB}}{(\ell \cdot p_i)^2} \right) \right]$$

This memory has no r-dependent $(e_1 g_2 - e_2 g_1)$ contribution, consistent with the angular momentum calculation.

Define $m_A = e_A + i\varepsilon_A{}^B e_B$, and we can write this as

$$\mathcal{E}_A = \sum_i Q_i e^{i\theta_i} \left(\frac{\ell_\mu \Delta J^\mu_{iA}}{\ell \cdot p_i} - \frac{(\ell \cdot \Delta p_i)\ell_\mu J^\mu_{iA}}{(\ell \cdot p_i)^2} \right)$$

Gravitational memory is work in progress.

Summary

- Classical observables in **dual** theories can be computed directly from amplitudes: no EOM needed.
- Observables from duality-rotated 3pt amplitudes + factorization: memory, waveform, impulse, angular impulse.
- Late-time log terms have trivial dyonic/NUT contribution.
- We derived classical duality-covariant E&M observables from KMOC.
- Memory derived directly from duality-rotated on-shell amplitudes. Constant contribution to memory at subleading order, from ΔJ.

- How to understand Taub-NUT memory from Amplitudes?
- How to dualise the Compton amplitude? Not possible in GR?
- Spinning soft waveforms and memory? Use Newman-Janis:

$$\mathcal{A}_3 \to \mathcal{A}_3 e^{i\theta + a\cdot q}.$$

- Duality-rotated waveform double copy? Dual-memory double copy?

Thank you for your attention.

Any questions?