Hilbert space and Unitarity in celestial and carrollian holography

based on 2411.19219 with L. Iacobacci and 2504.10577 with S. Agrawal

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Motivation

What is celestial holography good for?

It should provide a predictive, constraining framework for (gravitational) scattering

[See Shiraz'vision]

Important fundamental questions

What is the Hilbert space in celestial and carrollian holography?

- What is the symmetry group? Is it spontaneously broken?
- Which representations do states belong to? Are they unitary?

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- Which representations do states belong to? Are they unitary?

There are confusing statements in the literature... including:

- bulk and boundary Hilbert space are different
- Non-unitary
- Problem with translations (shift scaling dimension)
- Undesirable (distributional) two-point functions
- ...

Unitarity?

It is often said that CCFT is non-unitary, mainly because $\Delta \in 1 + i\mathbb{R}$.

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Certainly, celestial states should belong to UIRs of the Lorentz group.

Proof

$$U(a,\Lambda)^{-1} = U(a,\Lambda)^{\dagger} \qquad \Rightarrow \qquad U(0,\Lambda)^{-1} = U(0,\Lambda)^{\dagger} \, .$$

The paradox comes from an abuse of language:
States with $\Delta > 0$ are unitary w.r.t. $SO(2,2)$, *reflection-positive* w.r.t. $SO(1,3)$.

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Unitary irreps of the Lorentz group

Elementary representations [Dobrev-Mack-Petkova-Petrova-Todorov '77]

We start from the generators $J_{\mu\nu}$ of $SO(1,3)$ and split the indices,

$$J_{ij}\,, \quad J_{i0} = -\frac{P_i + K_i}{2}\,, \quad J_{i3} = \frac{P_i - K_i}{2}\,, \quad J_{03} = -D\,,$$

Elementary representations are characterized by states $|\Delta, \vec{x}\rangle_{J=\pm s}$ such that

$$\begin{aligned}
P_i\,|\Delta,\vec{x}\rangle &= -i\partial_i\,|\Delta,\vec{x}\rangle\,,\\
J_{ij}\,|\Delta,\vec{x}\rangle &= -i\left(x_i\partial_j - x_j\partial_i + iJ\epsilon_{ij}\right)|\Delta,\vec{x}\rangle\,,\\
D\,|\Delta,\vec{x}\rangle &= i\left(\Delta + x^i\partial_i\right)|\Delta,\vec{x}\rangle\,,\\
K_i\,|\Delta,\vec{x}\rangle &= i\left(2x_i\Delta + 2x_ix^j\partial_j - x^2\partial_i + 2iJx^j\epsilon_{ij}\right)|\Delta,\vec{x}\rangle\,.
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Unitarity can be achieved for

- $\Delta = 1 + i\mathbb{R}$ (continuous principal series)
- $\Delta \in (0,2)$ for $s = 0$ (complementary series)
- $2 - \Delta \in \mathbb{N}$ for $s = 0$ (exceptional discrete series)

Unitary Hilbert space for $\Delta \in 1 + i\mathbb{R}$

We rename $|\nu, \vec{x}\rangle_J \equiv |\Delta, \vec{x}\rangle_J$ for $\Delta = 1 + i\nu$. Then a generic ray in the Hilbert space is of the form

$$|\psi\rangle = \sum_{J=\pm s} \int_{-\infty}^{\infty} d\nu\, \rho(\nu) \int d^2\vec{x}\, \psi_J(\nu, \vec{x}) |\nu, \vec{x}\rangle_J \,,$$

with fixed $\rho(\nu)$ and invariant inner product

$$\langle\phi|\psi\rangle = \sum_{J=\pm s} \int_{-\infty}^{\infty} d\nu\, \rho(\nu) \int d^2\vec{x}\, \phi_J(\nu, \vec{x})^* \, \psi_J(\nu, \vec{x}) \,.$$

Allowed states have finite positive norm $||\psi|| = \sqrt{\langle\psi|\psi\rangle} < \infty$.

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Physicists will write

$${}_{J'}\langle \nu', \vec{y} \,|\, \nu, \vec{x} \rangle_J = \frac{\delta(\nu - \nu')\delta(\vec{x} - \vec{y})}{\rho(\nu)}\, \delta_{JJ'} \,,$$

interpreted as a (distributional) two-point function.

Complementary and discrete series

For $\Delta \in (0,2)$, we have

$$\langle \Delta, \vec{x}_1 \,| \Delta, \vec{x}_2 \rangle = |\vec{x}_{12}|^{-2\Delta} \,.$$

For $2 - \Delta \in \mathbb{N}$, we have

$$\langle \Delta, \vec{x}_1 | \Delta, \vec{x}_2 \rangle = |\vec{x}_{12}|^{-2\Delta} \ln(\mu |\vec{x}_{12}|) \,.$$

The logarithmic two-point function is part of the Hilbert space structure!

Celestial decomposition of scattering states

Poincaré algebra

The standard generators $\tilde{J}_{\mu\nu}, \tilde{P}_\mu$ satisfy

$$\left[\tilde{J}_{\mu\nu}\,, \tilde{J}_{\rho\sigma}\right] = -i\left(\eta_{\mu\rho}\,\tilde{J}_{\nu\sigma} + \eta_{\nu\sigma}\,\tilde{J}_{\mu\rho} - \eta_{\mu\sigma}\,\tilde{J}_{\nu\rho} - \eta_{\nu\rho}\,\tilde{J}_{\mu\sigma}\right)\,,$$

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We split the indices and write

$$\tilde{J}_{ij} = J_{ij}\,, \quad \tilde{J}_{i0} = -\frac{P_i + K_i}{2}\,, \quad \tilde{J}_{i3} = \frac{P_i - K_i}{2}\,, \quad \tilde{J}_{03} = -D\,,$$

and

$$\tilde{P}_0 = \frac{H + K}{\sqrt{2}}\,, \qquad \tilde{P}_i = -\sqrt{2}\,B_i\,, \qquad \tilde{P}_3 = \frac{K - H}{\sqrt{2}}\,.$$

Massless particles

Let's explicitly perform Wigner's construction of massless particle states. We pick a reference massless momentum $k^\mu = (1, 0^i, 1)$, left invariant by the little group generated by $\langle J_{ij}, K_i \rangle$. We specify a representation

$$J_{ij}|k,J\rangle = J\varepsilon_{ij}|k,J\rangle\,, \qquad K_i|k,J\rangle = 0\,, \qquad \tilde{P}_\mu|k,J\rangle = k_\mu|k,J\rangle\,,$$

which we then boost to an arbitrary frame [KN-West '23]

$$|p(\omega,\vec{x}),J\rangle \equiv e^{ix^i P_i}\, e^{i\ln\omega\, D}|k,J\rangle\,.$$

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Using the algebra, we can check that

$$\tilde{P}_\mu|p, J\rangle = p_\mu|p, J\rangle\,, \qquad p^\mu(\omega, \vec{x}) = \omega\,(1 + x^2, 2\vec{x}, 1 - x^2)\,.$$

The action of the Lorentz generators is

$$
\begin{aligned}
P_i \left| p \right\rangle &= -i\partial_i \left| p \right\rangle , \\
J_{ij} \left| p \right\rangle &= -i \left(x_i \partial_j - x_j \partial_i + iJ\epsilon_{ij} \right) \left| p \right\rangle , \\
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The Hilbert space contains states of the form

$$|\psi\rangle = \sum_{J=\pm s} \int [d^3 p(\omega, \vec{x})] \, \psi_J(\omega, \vec{x}) |p(\omega, \vec{x})\rangle_J \,,$$

with $[d^3 p(\omega, \vec{x})] = \omega \, d\omega \, d^2 \vec{x}$ and inner product

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$$

Finite norm requires

$$
\psi_J(\omega,\vec{x}) = o\left(\omega^{-1}\right) , \qquad (\omega\to 0) .
$$

Branching down to Lorentz

Main question

How do massless $ISO(1,3)$ irreps decompose w.r.t. $SO(1,3)$?

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The answer is given by the familiar Mellin transform:

[Mukunda '68, de Boer-Solodukhin '03, Cheung-Fuente-Sundrum '16, Pasterski-Shao '17]

$$\psi_J(\Delta,\vec{x}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega\, \omega^{\Delta-1}\, \psi_J(\omega,\vec{x})\,.$$

This is holomorphic on the complex half-plane Re$(\Delta) \geq 1$. The inverse transform

$$\psi_J(\omega,\vec{x}) = \frac{1}{\sqrt{2\pi i}} \int_{c-i\infty}^{c+i\infty} d\Delta\, \omega^{-\Delta}\, \psi_J(\Delta,\vec{x})\,,$$

is valid when no singularity lies to the right of the integration path.

The bordeline choice $c = 1$ yields a decomposition over the principal series!

A comment about translations

An apparent paradox

It seems that translations take states outside the principal series:

$$\tilde{P}^0|\Delta,\vec{0}\,\rangle = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega\,\omega^{\Delta}|p(\omega,\vec{0})\rangle = |\Delta+1,\vec{0}\,\rangle\,.$$

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They would violate unitarity.

Group theory immediately tells us that this cannot be correct.

Proper treatment

$$e^{-ia^\mu\tilde{P}_\mu}|\nu,\vec{x}\rangle = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega\,\omega^{i\nu}e^{-i\omega a^\mu q_\mu(\vec{x})}|p(\omega,\vec{x})\rangle$$
$$= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^\infty d\nu'\,\frac{\Gamma(\epsilon+i(\nu-\nu'))}{(ia^\mu q_\mu(\vec{x})+\epsilon)^{i(\nu-\nu')}}\int_0^\infty d\omega\,\omega^{i\nu'}|p(\omega,\vec{x})\rangle$$
$$= \frac{1}{2\pi}\int_{-\infty}^\infty d\nu'\,\frac{\Gamma(\epsilon+i(\nu-\nu'))}{(ia^\mu q_\mu(\vec{x})+\epsilon)^{i(\nu-\nu')}}\,|\nu',\vec{x}\rangle$$

Massive scalar particles

Harmonic analysis

Simply put, we want to decompose a generic normalizable spinning wavefunction $\psi_\sigma(\hat{p})$ over the mass-shell $\hat{p} \in \mathbb{H}^3$ onto a basis of functions which provide a realization of the Lorentz UIRs.

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The boundary-bulk propagators provide this basis,

$$G_\nu(\hat{p};\vec{x}) = \frac{1}{(-\hat{p}\cdot q(\vec{x}))^{\Delta_\nu}}\,, \qquad \Delta_\nu = 1 + i\nu\,.$$

We can check that they satisfy the Casimir condition

$$J_{\mu\nu}J^{\mu\nu}G_\nu(\hat{p};\vec{x}) = \Delta_{\mathbb{H}}\,G_\nu(\hat{p};\vec{x}) = [\Delta_\nu(\Delta_\nu - d)]\,G_\nu(\hat{p};\vec{x})\,, \quad \forall \vec{x} \in \mathbb{R}^2\,.$$

Note: $\vec{x}$ is the analogue of angular momentum $m = -\ell, ..., \ell$ for spherical harmonics.

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Result [MacDowell-Roskies '72, de Boer-Solodukhin '03, Pasterski-Shao-Strominger '17]

$$|p\rangle = \int_0^\infty d\nu\,\mu(\nu)\int d^2\vec{x}\,G_{-\nu}(\hat{p};\vec{x})\,|\nu,\vec{x}\rangle\,, \qquad \mu(\nu) = \text{Plancherel measure}$$

Massive spinning particles [Iacobacci-KN '24]

Closely related to massive primary wavefunctions: [Law-Zlotnikov '20, Iacobacci-Muck '20]

Main technicality

Massive particles carry $SO(3)$-spin, but continuous principal series carry $SO(2)$-spin. Part of the harmonic decomposition thus features

$$V_s^{\mathrm{SO}(3)} = \bigoplus_{\ell=0}^{s} V_\ell^{\mathrm{SO}(2)}$$

We did half-integer spin in arbitrary dimension. This is highly technical, fortunately we could recycle formulae from [Costa-Goncalves-Penedones '14, Iacobacci-Muck '20].

BMS symmetries and spontaneous breaking

BMS supertranslations

BMS group

$$\mathrm{BMS}_4 = \mathrm{SO}(1,3) \ltimes \mathcal{E}_{-1}(S^2)\,.$$

Supertranslation charges can be written

$$Q_T(u) = \int_{S^2_u} d^2\vec{x}\, T(\vec{x})\, \mathcal{M}(u,\vec{x})\,,$$

with the covariant Bondi mass aspect satisfying the current non-conservation

$$\partial_u \mathcal{M} = \frac{1}{4}\left(\partial^i \partial^j N_{ij} + C^{ij}\partial_u N_{ij}\right) - \kappa^2\, T_{uu} \overset{u\to-\infty}{\sim} 0\,.$$

such that $Q_T(u)$ is the canonical generator only near spatial infinity.

BMS Ward identities

Reminder: carrollian conformal fields = massless particles

$$O_{\Delta,J}(u,\vec{x}) = \int_0^\infty d\omega\, \omega^{\Delta-1} e^{i\omega u} a_J^\dagger(p(\omega,\vec{x}))\,,$$

Supertranslations act on the carrollian fields by

$$[Q_T\,, O_{\Delta,J}(\mathbf{x})] = i\, T(\vec{x})\, \partial_u O_{\Delta,J}(\mathbf{x})\,.$$

From this we can derive the Ward identity

$$\langle 0| Q_T\, O_1(\mathbf{x}_1) \ldots O_n(\mathbf{x}_n) - O_1(\mathbf{x}_1) \ldots O_n(\mathbf{x}_n)\, Q_T |0\rangle$$
$$= i \sum_{a=1}^{n} T(\vec{x}_a) \partial_{u_a} \langle 0| O_1(\mathbf{x}_1) \ldots O_n(\mathbf{x}_n) |0\rangle\,.$$

If the RHS is nonzero (order parameter), we will be forced to conclude SSB

$$Q_T |0\rangle \neq 0\,.$$

Soft graviton theorem implies SSB

We assume the validity of Weinberg's soft theorem,

$$\lim_{\omega\to 0}\omega\, S_{n+1}\left(p_1\,,\ldots,p_n\,;\omega q(\vec{y}),\varepsilon_i\right)=\kappa\sum_{a=1}^{n}\frac{\left(p_a\cdot\varepsilon_i(\vec{y})\right)^2}{p_a\cdot q(\vec{y})}\,S_n\left(p_1\,,\ldots,p_n\right)$$
$$=\kappa\sum_{a=1}^{n}\omega_a\,\frac{4(y^i-x_a^i)^2}{|\vec{y}-\vec{x}_a|^2}\,S_n\left(p_1\,,\ldots,p_n\right)\,.$$

We transform the n hard legs to carrollian basis,

$$\lim_{\omega\to 0}\omega\Big[\langle O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)a_i^\dagger(\omega q(\vec{y}))\rangle-\langle a_i(\omega q(\vec{y}))O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\rangle\Big]$$
$$=-i\kappa\sum_{a=1}^{n}\frac{2(y^i-x_a^i)^2}{|\vec{y}-\vec{x}_a|^2}\partial_{u_a}\langle O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)\rangle\,.$$

This is the Ward identity with

$$T(\vec{x}_a;\vec{y},\varepsilon_i)=\frac{2(y^i-x_a^i)^2}{|\vec{y}-\vec{x}_a|^2}\,,\qquad Q_T|0\rangle\equiv\lim_{\omega\to 0}\omega a_i^\dagger(\omega q(\vec{y}))|0\rangle\,.$$

Carrollian Goldstone theorem

Spontaneous symmetry breaking is characterised by

$$\langle 0|[Q_T\,, O]|0\rangle \neq 0\,,$$

where in this case $O = O_1(\mathbf{x}_1)\ldots O_n(\mathbf{x}_n)$. We insert a resolution of the identity on the Hilbert space,

$$\langle 0|[Q_T\,, O]|0\rangle = \sum_n \left[\langle 0|Q_T|n\rangle\langle n|O|0\rangle - \langle 0|O|n\rangle\langle n|Q_T|0\rangle\right]\,,$$

and there must exist at least one particle species $|G\rangle$ such that

$$\langle 0|Q_T|G\rangle \neq 0\,, \qquad \leftrightarrow \qquad \lim_{u\to-\infty}\langle 0|\mathcal{M}(u,\vec{x})|G\rangle \neq 0\,.$$

Carrollian Goldstone theorem

But current conservation near spatial infinity requires

$$0 = \langle 0|\partial_u \mathcal{M}(u,\vec{x})|G\rangle = \partial_u \langle 0|e^{-iuH}\mathcal{M}(0,\vec{x})\, e^{iuH}|G\rangle\Big|_{u\to-\infty}$$

Hence the Goldstone particle cannot be one of Wigner's particles,

$$H|p(\omega,\vec{x})\rangle = \omega|p(\omega,\vec{x})\rangle \neq 0\,.$$

Instead it is a simple Lorentz UIR in the exceptional discrete series:

$$|\Delta,\vec{x}\,\rangle_G = G_\Delta(\vec{x})|0\rangle\,, \qquad \Delta = -1\,.$$

The corresponding two-point function,

$${}_G\langle \Delta,\vec{x}_1|\Delta,\vec{x}_2\rangle_G = \mathcal{N}\,|\vec{x}_{12}|^{-2\Delta}\ln(\mu|\vec{x}_{12}|)\,,$$

directly agrees with earlier discussion of IR divergences!

[Nande-Pate-Strominger '17, Himwich-Narayanan-Pate-Paul-Strominger '20]

Lessons

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Main message

Group theory provides very rigid mathematical structures for CCFT!

We have learned that

- Unitarity is well and alive
- Celestial primaries with $\Delta \in 1 + i\mathbb{R}$ describe scattering states
- Celestial primaries with $\Delta = 0, -1, ..$ describe Goldstone/soft modes
  They *necessarily* have logarithmic two-point functions
- Supertranslations are spontaneously broken

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  They *necessarily* have logarithmic two-point functions
- Supertranslations are spontaneously broken

Warning

Standard CFT does not apply!
In CCFT we have *unitarity* rather than *reflection-positivity*!

A general framework is still lacking. A good starting point would be a rigorous conformal block expansion [Gadde'17].