Quantum Cluster Algebras & Representations of Shifted Quantum Affine Algebras

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CONTEXT:

- $\mathfrak{g}$ simple, f.d. Lie algebra$/\mathbb{C}$, type ADE.
- For every μ integral weight of $\mathfrak{g}$, the shifted quantum affine algebra $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ is defined (Finkelberg – Tsymbaliuk)
- To study its representation theory, we can look at $\mathcal{O}^\mu$ (Hernandez)

  and moreover at $\mathcal{O}^{sh} := \bigoplus_\mu \mathcal{O}^\mu$
- Thanks to a fusion product in $\mathcal{O}^{sh}$, we can consider the Grothendieck ring $K(\mathcal{O}^{sh})$.
- There is an injective ring morphism:

$$K(\mathcal{O}^{b,\pm}) \hookrightarrow K(\mathcal{O}^{sh}).$$

Thm (Geiss-Hernandez-Leclerc)

The Grothendieck ring $K(\mathcal{O}_{\mathbb{Z}}^{sh})$ has a cluster algebra structure. Namely: there is a cluster algebra $\mathcal{A}$ s.t. $\mathcal{A} \hookrightarrow K(\mathcal{O}_{\mathbb{Z}}^{sh})$ and $\widehat{\mathrm{Im}}(\mathcal{A}) \simeq K(\mathcal{O}_{\mathbb{Z}}^{sh})$.

example type A_1

$$\begin{array}{ccc}
\mathcal{A} & & K(\mathcal{O}_{\mathbb{Z}}^{sh}) \\
\vdots & & \vdots \\
x_4 & \longmapsto & [L_4^+] \\
x_2 & \longmapsto & [L_2^+] \\
x_0 & \longmapsto & [L_0^+] \\
x_{-2} & \longmapsto & [L_{-2}^-] \\
x_{-4} & \longmapsto & [L_{-4}^-] \\
x_{-6} & \longmapsto & [L_{-6}^-] \\
\vdots & & \vdots
\end{array}$$

Exchange relation

$$(x_0 x_0' = 1 + x_2 x_{-2}) \longmapsto [L_0^+][L_0^-] = 1 + [L_2^+][L_2^-]$$

QQ - system

Thm (P.)

There exists a quantum cluster algebra structure $\mathcal{A}_t$ for the GHL's cluster algebra.

Let us call $K_t(\mathcal{O}^{sh}_{\mathbb{Z}})$ the completion of $\mathcal{A}_t$ and we refer to it as the quantum Grothendieck ring of $\mathcal{O}^{sh}_{\mathbb{Z}}$ (inspired by Varagnolo-Vasserot, Nakajima, Hernandez, Bittmann)

We have $$\underbrace{K_t(\mathcal{O}^{b,\pm}_{\mathbb{Z}})}_{U_q(\hat{b})\text{-modules}} \hookrightarrow K_t(\mathcal{O}^{sh}_{\mathbb{Z}})$$

We get quantum QQ-systems as quantum exchange relations.

Thm (P.)

In type A_1 we have the following phenomenon:

- $\mathcal{U}^{\pm}_{t,loc}(\mathfrak{sl}_2)$ has a quantum cluster algebra structure.
- $\mathcal{U}^{\pm}_{t,loc}(\mathfrak{sl}_2)$ is isomorphic to $\mathbb{C}_t[SL_2^{w_0,w_0}]$.

COR.

$\mathcal{A}_t$ contains infinitely many copies of $\mathcal{U}^{\pm}_{t,loc}(\mathfrak{sl}_2)$.

CONCLUSION / QUESTIONS

1. We now have a framework to develop a (q,t)-character theory (as for $\mathcal{U}_q(\hat{\mathfrak{g}})$) and hopefully compute many q-characters.
2. We can compare our construction with the one of Qin: are they the same?
3. Is there an analogue result about q-oscillator algebras in other types?

THANK

YOU!