Crystals and quantum twist automorphisms

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New perspectives in quantum representation theory

(arXiv:2507.01306 with Woo-Seok Jung)

Motivation

- $\mathrm{A} = (a_{i,j})_{i,j \in I}$ = generalized symmetrizable Cartan matrix,
- $U_q(\mathfrak{g})$ = quantum group associated with A,
- $\mathrm{A}_q(\mathfrak{n})$ = quantum unipotent coordinate ring, which is $\simeq (U_q^-(\mathfrak{g}))^\vee$,

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- $\mathrm{G}^{\mathrm{up}}(\infty)$ = upper global basis (or dual canonical basis) of $\mathrm{A}_q(\mathfrak{n})$,
- $B(\infty)$ = crystal basis of $U_q^-(\mathfrak{g})$,

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$$B(\infty) \xleftrightarrow{1-1} \mathrm{G}^{\mathrm{up}}(\infty)$$

- R = quiver Hecke algebra (or KLR algebra) associated with A,
- R-gmod = finite-dimensional graded R-module category,

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- $\mathrm{A}_q(\mathfrak{n}(w))$ has a quantum cluster algebra structure.

([Geiß-Leclerc-Schröer 1], [Goodearl-Yakimov])

Cluster alg str : $\{\text{cluster monomials}\} \hookrightarrow \mathrm{G}^{\mathrm{up}}(w)$

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- $^{\exists}$ subcategory $\mathcal{C}_w$ of R-gmod such that $K_q(\mathcal{C}_w) \simeq \mathrm{A}_q(\mathfrak{n}(w))$.

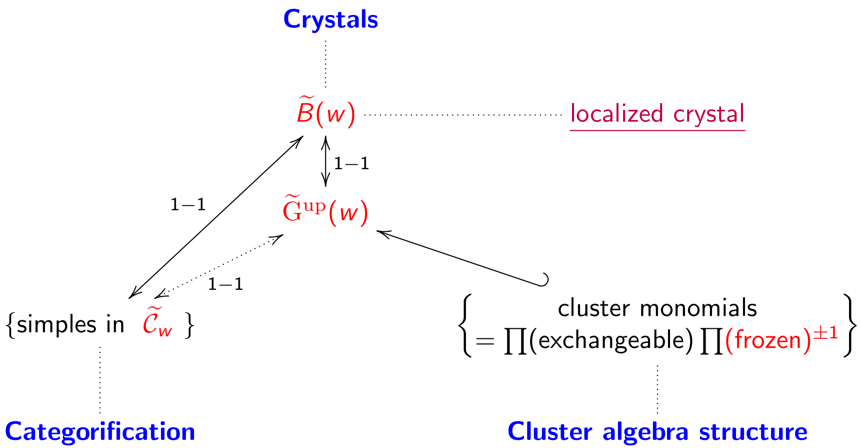
Categorification : $\{\text{simples in } \mathcal{C}_w\} \xleftrightarrow{1-1} B(w)$

([Varagnolo-Vasserot]) ([Lauda-Vazirani])

Three structures related to $A_q(\mathfrak{n}(w))$

$$
\begin{tikzcd}
 & \textbf{Crystals} \arrow[d, dash, dotted] & \\
 & B(w) \arrow[d, leftrightarrow, "1-1"] \arrow[dl, leftrightarrow, "1-1"'] & \\
 & \mathrm{G}^{\mathrm{up}}(w) \arrow[dl, leftrightarrow, dotted, "1-1"] & \\
\{\text{simples in } \mathcal{C}_w\} \arrow[d, dash, dotted] & & \left\{ \begin{array}{c} \text{cluster monomials} \\ = \prod(\text{exchangeable}) \prod(\text{frozen}) \end{array} \right\} \arrow[ul, hook] \arrow[d, dash, dotted] \\
\textbf{Categorification} & & \textbf{Cluster algebra structure}
\end{tikzcd}
$$

Three structures related to the localized algebra $\widetilde{A}_q(\mathfrak{n}(w))$



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Goal: Interpret η_w in terms of crystals!

Localized crystals

- $A := (a_{ij})_{i,j\in I}$ (symmetrizable generalized Cartan matrix)
- $\Pi := \{\alpha_i \mid i \in I\}$ (simple roots)
- $\Pi^\vee := \{h_i \mid i \in I\}$ (simple coroots) $\quad (\langle h_i, \alpha_j\rangle = a_{i,j}$ for all $i, j)$
- $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (root lattice)
- $P :=$ weight lattice, $P^\vee :=$ dual weight lattice
- $\Lambda_i \in P^+$: fundamental weight (i.e., $\Lambda_i(h_j) = \delta_{ij}$)
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Definition (Quantum group)

(i) $U_q(\mathfrak{g})$:= the **k**-algebra generated by e_i, f_i $(i \in I)$ and q^h $(h \in \mathsf{P}^\vee)$ satisfying certain defining relations determined by C.

(ii) $U_q^-(\mathfrak{g})$:= the subalgebra of $U_q(\mathfrak{g})$ generated by f_i $(i \in I)$.

(iii) $\mathrm{A}_q(\mathfrak{n})$:= the dual of $U_q^-(\mathfrak{g})$ (quantum unipotent coordinate ring)

Definition (Crystals)

A $U_q(\mathfrak{g})$-crystal is a set B endowed with maps $\varphi_i, \varepsilon_i \colon B \to \mathbb{Z} \sqcup \{\infty\}$, $\mathrm{wt} \colon B \to \mathrm{P}$ and

$$\tilde{e}_i,\ \tilde{f}_i \colon B \to B \sqcup \{0\} \qquad \text{(for all } i \in I)$$

which satisfy the following conditions:

- ...
- for $b, b' \in B$ and $i \in I$, $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$,
- ...

crystal $B \rightsquigarrow$ crystal graph with arrows induced from $\tilde{f}_i$

vertices : B, arrows : $b \xrightarrow{i} b'$ iff $b' = \tilde{f}_i(b)$

Definition

(i) $B(\infty)$:= the crystal of the half $U_q^-(\mathfrak{g})$

(ii) $B(\Lambda)$:= the highest weight crystal for $\Lambda \in \mathsf{P}^+$

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Choose and fix $w \in \mathrm{W}$. We then define

- $\mathrm{A}_q(\mathfrak{n}(w)) =$ subalg of $\mathrm{A}_q(\mathfrak{n})$ gen. by dual PBW vectors w.r.t. w,
- $\mathrm{G}^{\mathrm{up}}(w) := \mathrm{G}^{\mathrm{up}}(\infty) \cap \mathrm{A}_q(\mathfrak{n}(w))$,
- $B(w) := \{b \in B(\infty) \mid \mathrm{G}^{\mathrm{up}}(b) \in \mathrm{G}^{\mathrm{up}}(w)\}$,

  where $\mathrm{G}^{\mathrm{up}}(b)$ is the member of $\mathrm{G}^{\mathrm{up}}(\infty)$ corresponding to b.

Definition (quantum unipotent minor)

$\mathrm{D}(w\Lambda, \Lambda) \in \mathrm{G}^{\mathrm{up}}(w)$ corresponding to the extremal vector $wu_\Lambda \in B(\Lambda)$.
(* u_Λ is the highest wight vector of $B(\Lambda)$).

Note $\mathrm{D}(w\Lambda_i, \Lambda_i)$ is frozen in $\mathrm{A}_q(\mathfrak{n}(w))$.

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Definition (Localized quantum unipotent coordinate ring)

$\widetilde{\mathrm{A}}_q(\mathfrak{n}(w)) :=$ localization of $\mathrm{A}_q(\mathfrak{n}(w))$ by $\mathrm{D}(w\Lambda_i, \Lambda_i)$ for $i \in I$.

$\widetilde{\mathrm{G}}^{\mathrm{up}}(w) :=$ upper global basis of $\widetilde{\mathrm{A}}_q(\mathfrak{n}(w))$.

For $i \in I$, $\mathsf{c}_i \in B(w)$ corresponding to the frozen $\mathrm{D}(w\Lambda_i, \Lambda_i)$.

Definition (Localized crystal)

$$\widetilde{B}(w) := B(w) \times \mathbb{Z}^{\oplus I} / \sim,$$

where $\sim$ is defined as follows:

let $(b, \mathbf{a}), (b', \mathbf{a}') \in B(w) \times \mathbb{Z}^{\oplus I}$ and write $\mathbf{a} = (a_i)_{i \in I}$ and $\mathbf{a}' = (a'_i)_{i \in I}$.
Define

$$(b, \mathbf{a}) \sim (b', \mathbf{a}')$$

if $^\exists$ a positive integer N such that $N + \min_{i \in I}\{a_i, a'_i\} \geq 0$ and

$$\mathrm{G}^{\mathrm{up}}(b) \prod_{i \in I} \mathrm{G}^{\mathrm{up}}(\mathsf{c}_i)^{N+a_i} \equiv_q \mathrm{G}^{\mathrm{up}}(b') \prod_{i \in I} \mathrm{G}^{\mathrm{up}}(\mathsf{c}_i)^{N+a'_i}.$$

Note

- $\widetilde{B}(w)$ has a crystal structure ([Nakashima], [Kashiwara-Nakashima]).

Crystal-theoretic analogue of η_w

Definition (Quantum twist automorphism)

η_w := the automorphism on the localized algebra $\widetilde{\mathrm{A}}_q(\mathfrak{n}(w))$ defined by

$$\eta_w(\mathrm{D}(v\Lambda,\Lambda)) \equiv_q \mathrm{D}(w\Lambda,\Lambda)^{-1}\mathrm{D}(w\Lambda,v\Lambda)$$

for any $v \in \mathsf{W}$ with $v \leq w$ and $\Lambda \in \mathsf{P}^+$.

Proposition ([Kimura-Oya]) η_w permutes $\widetilde{\mathrm{G}}^{\mathrm{up}}(w)$.

Let R = quiver Hecke alg, $R\text{-gmod}$ = cat. of f. d. Rmodules

Definition (Categorification using quiver Hecke algebras)

$\mathcal{C}_w$:= full subcat of $R\text{-gmod}$ consisting of modules M s.t.

$$\mathrm{W}(M) \subset \mathrm{span}_{\mathbb{R}_{\geq 0}}(\Delta^+ \cap w\Delta^-),$$

where $\mathrm{W}(M) = \{\gamma \in \mathrm{Q}_+ \cap (\beta - \mathrm{Q}_+) \mid e(\gamma, \beta - \gamma)M \neq 0\}$, Δ^+ is the set of positive roots, and $\Delta^- = -\Delta^+$.

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Definition (Localized category)

$\widetilde{\mathcal{C}}_w$:= the localization of $\mathcal{C}_w$ by simples corr. to $\mathrm{D}(w\Lambda_i, \Lambda_i)$ for $i \in I$

Then we have

(i) $K_q(\widetilde{\mathcal{C}}_w) \simeq \widetilde{\mathrm{A}}_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{n}(w))$.

(ii) $\widetilde{\mathcal{C}}_w$ is a rigid monoidal category, i.e., $\mathscr{D}_w :=$ right dual functor of $\widetilde{\mathcal{C}}_w$.

(iii) $\mathscr{D}_w$ permutes $\widetilde{\mathbf{G}}(w) := \{$ simple objects in $\widetilde{\mathcal{C}}_w\}$.

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Note Ishibashi-Oya show that

$$\eta_w^{-1} = [\mathscr{D}_w] \qquad \text{on the bases } \widetilde{\mathrm{G}}^{\mathrm{up}}(w) \text{ and } \widetilde{\mathbf{G}}(w).$$

Definition (Crystal-theoretic analogue of η_w)

The permutation

$$\mathcal{D}_w : \widetilde{B}(w) \to \widetilde{B}(w)$$

is defined to commute the following:

$$\begin{tikzcd}
\widetilde{\mathbf{G}}(w) \arrow[d, "{[\mathscr{D}_w]}"'] & \widetilde{B}(w) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\mathcal{D}_w"'] & \widetilde{\mathrm{G}}^{\mathrm{up}}(w) \arrow[d, "\eta_w^{-1}"'] \\
\widetilde{\mathbf{G}}(w) & \widetilde{B}(w) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \widetilde{\mathrm{G}}^{\mathrm{up}}(w)
\end{tikzcd}.$$

Categorification Localized crystal Quantum coor. ring

Two crystal parameterizations of $\widetilde{B}(w)$: PBW and String

Let $\mathbf{i} := (i_1, i_2, \dots, i_m) \in R(w)$ be a reduced expression of w.

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PBW parametrizations

(i) For any $b \in B(w)$, define

$$\mathrm{PBW}_{\mathbf{i}}(b) := \mathbf{a} = (a_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^m$$

such that $\mathrm{G}^{\mathrm{up}}(b) \equiv_{q=0}$ dual PBW monomial $F^{\mathrm{up}}(\mathbf{a})$ w.r.t. $\mathbf{i}$.

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(ii) We have $B(w) \xrightarrow{1-1} \mathsf{P}_{\mathbf{i}}(w)$, where

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(iii) We have $\mathfrak{P}_{\mathbf{i}} : \widetilde{B}(w) \xrightarrow{1-1} \widetilde{\mathsf{P}}_{\mathbf{i}}(w)$, where

$$\widetilde{\mathsf{P}}_{\mathbf{i}}(w) := \mathsf{P}_{\mathbf{i}}(w) + \sum_{k \in I} \mathbb{Z}(\mathrm{PBW}_{\mathbf{i}}(\mathsf{c}_k)) = \mathbb{Z}^m,$$

String parametrizations

(i) For any $b \in B(w)$, define

$$\mathrm{STR}_{\mathbf{i}}(b) := (t_k)_{k \in [1,m]},$$

where $t_1 = \varepsilon_{i_1}(b)$, $t_2 = \varepsilon_{i_2}(\tilde{e}_{i_1}^{t_1}(b))$, ..., $t_m = \varepsilon_{i_m}(\tilde{e}_{i_{m-1}}^{t_{m-1}} \cdots \tilde{e}_{i_2}^{t_2} \tilde{e}_{i_1}^{t_1}(b))$.

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Main result

For $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in R(w)$, we have two parameterizations:

(i) **PBW parametrization** $\mathfrak{P}_{\mathbf{i}} : \ \widetilde{B}(w) \xrightarrow{1-1} \widetilde{\mathsf{P}}_{\mathbf{i}}(w)$,

(ii) **String parametrization** $\mathfrak{S}_{\mathbf{i}} : \ \widetilde{B}(w) \xrightarrow{1-1} \widetilde{\mathsf{S}}_{\mathbf{i}}(w)$,

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(ii) **String parametrization** $\mathfrak{S}_\mathbf{i} : \ \widetilde{B}(w) \xrightarrow{1-1} \widetilde{\mathsf{S}}_\mathbf{i}(w)$,

(iii) The bijection $\psi_\mathbf{i} := \mathfrak{S}_\mathbf{i} \circ \mathfrak{P}_\mathbf{i}^{-1}$, i.e.,

$$\begin{tikzcd}
 & \widetilde{B}(w) \arrow[dl, "\mathfrak{P}_\mathbf{i}"'] \arrow[dr, "\mathfrak{S}_\mathbf{i}"] & \\
\mathbb{Z}^m = \widetilde{\mathsf{P}}_\mathbf{i}(w) \arrow[rr, "\psi_\mathbf{i}"] & & \widetilde{\mathsf{S}}_\mathbf{i}(w) = \mathbb{Z}^m
\end{tikzcd}$$

Recall $\mathcal{D}_w : \widetilde{B}(w) \xrightarrow{1-1} \widetilde{B}(w)$ (crystal-theoretic analogue of η_w)

Theorem ([Jung-P.]) For any $b \in \widetilde{B}(w)$,

(i) $\mathrm{PBW}_{\mathbf{i}}(\mathcal{D}_w(b)) = \psi_{\mathbf{i}}^{-1} \circ \mathcal{M}_{\mathbf{i}} \circ \mathcal{N}_{\mathbf{i}}(-\mathrm{PBW}_{\mathbf{i}}(b))$,

(ii) $\mathrm{STR}_{\mathbf{i}}(\mathcal{D}_w(b)) = -\mathcal{M}_{\mathbf{i}} \circ \mathcal{N}_{\mathbf{i}} \circ \psi_{\mathbf{i}}^{-1}(\mathrm{STR}_{\mathbf{i}}(b))$,

where $m \times m$ matrices $N_{\mathbf{i}} = (n_{p,q})_{p,q\in[1,m]}$, $M_{\mathbf{i}} = (m_{p,q})_{p,q\in[1,m]}$ defined by

$$n_{p,q} = \begin{cases} 1 & \text{if } p = q, \\ -1 & \text{if } p^+ = q, \\ 0 & \text{otherwise,} \end{cases} \qquad m_{p,q} = \begin{cases} \langle h_{i_p}, s_{i_{p+1}} s_{i_{p+2}} \cdots s_{i_q} \Lambda_{i_q} \rangle & \text{if } p \le q, \\ 0 & \text{otherwise,} \end{cases}$$

where $k^+ := \min(\{m+1\} \cup \{k+1 \le j \le m \mid i_j = i_k\}$

Remark

- When it is of finite type, $\psi_{\mathbf{i}}$ can be computed by using SageMath computer program. Thus, the $\mathcal{D}_w$ can be computed by using SageMath.
- We obtain similar formulas in terms of g-vectors.
- The theorem gives an answer to the problem by Nakashima:

"Describe $\mathscr{D}_w(M)$ explicitly in terms of crystals."

(Idea of proof)

(a) Cosider two kinds of g-vectors $\mathrm{g}^{\mathrm{L}}_{\mathbf{i}}(b)$ and $\mathrm{g}^{\mathrm{R}}_{\mathbf{i}}(b)$ for $b \in \widetilde{B}(w)$

(b) ([Kashiwara-Kim], [Kashiwara-Kim-Oh-P.])

$$\mathrm{g}^{\mathrm{R}}_{\mathbf{i}}(M) + \mathrm{g}^{\mathrm{L}}_{\mathbf{i}}(\mathscr{D}_w(M)) = 0 \qquad \text{for a simple module } M \in \mathcal{C}_w.$$

(c) Consider the following commuting diagram:

$$\begin{tikzcd}
 & & \widetilde{B}(w) \arrow[dl, "\mathfrak{P}_{\mathbf{i}}"'] \arrow[dr, "\mathfrak{S}_{\mathbf{i}}"] & & \\
\mathrm{G}^{\mathrm{R}}_{\mathbf{i}}(w) & \widetilde{\mathrm{P}}_{\mathbf{i}}(w) \arrow[l, "\mathcal{N}_{\mathbf{i}}"', "{[KK]}"] \arrow[rr, "\psi_{\mathbf{i}}"] & & \widetilde{\mathrm{S}}_{\mathbf{i}}(w) & \mathrm{G}^{\mathrm{L}}_{\mathbf{i}}(w), \arrow[l, "\mathcal{M}_{\mathbf{i}}"', "{[FO]}"]
\end{tikzcd}$$

where [KK] = [Kashiwara-Kim] and [FO] = [Fujita-Oya].

Example

Type A_2, $w = w_0$, $\mathbf{i} = (1,2,1) \in R(w_0)$, i.e., $B(\infty) = B(w_0)$.

By direct computation,

$$\mathsf{c}_1 = \tilde{f}_2 \tilde{f}_1 \mathbf{1} \in B(\infty) \quad \text{and} \quad \mathsf{c}_2 = \tilde{f}_1 \tilde{f}_2 \mathbf{1} \in B(\infty)$$

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PBW parametrization

- $B(\infty) \overset{1-1}{\longleftrightarrow} \mathrm{P}_{\mathbf{i}}(w_0) = \{(x,y,z) \mid x,y,z \geq 0\} = \mathbb{Z}^3_{\geq 0}$.

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- $B(\infty) \xleftrightarrow{1-1} \mathrm{P}_{\mathbf{i}}(w_0) = \{(x,y,z) \mid x,y,z \geq 0\} = \mathbb{Z}^3_{\geq 0}$.
- $P_1 := \mathrm{PBW}_{\mathbf{i}}(\mathrm{c}_1) = (1,0,1)$ and $P_2 := \mathrm{PBW}_{\mathbf{i}}(\mathrm{c}_2) = (0,1,0)$ imply

$$\widetilde{B}(\infty) \xleftrightarrow{1-1} \widetilde{\mathrm{P}}_{\mathbf{i}}(w_0) = \mathrm{P}_{\mathbf{i}}(w_0) + \mathbb{Z}P_1 + \mathbb{Z}P_2 = \mathbb{Z}^3.$$

String parametrization

- $B(\infty) \overset{1-1}{\longleftrightarrow} \mathsf{S}_{\mathbf{i}}(w_0) = \{(a,b,c) \mid a \geq 0, b \geq c \geq 0\} \subset \mathbb{Z}^3_{\geq 0}$.

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- $S_1 := \mathrm{STR}_{\mathbf{i}}(\mathsf{c}_1) = (0, 1, 1)$ and $S_2 := \mathrm{STR}_{\mathbf{i}}(\mathsf{c}_2) = (1, 1, 0)$ imply

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The bijection $\psi_{\mathbf{i}} : \widetilde{\mathsf{P}}_{\mathbf{i}}(w_0) \longrightarrow \widetilde{\mathsf{S}}_{\mathbf{i}}(w_0)$

$$\psi_{\mathbf{i}}(x,y,z) = (x + y - \min\{x,z\}, y + z, \min\{x,z\}).$$

By a direct computation, we have

$$N_{\mathbf{i}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{\mathbf{i}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Computation for $\mathcal{D}^{-1}$

For $\mathrm{PBW}_{\mathbf{i}}(b) = (x, y, z)$, we have

$$\begin{aligned} \mathrm{PBW}_{\mathbf{i}}(\mathcal{D}^{-1}(b)) &= -\mathcal{N}_{\mathbf{i}}^{-1} \circ \mathcal{M}_{\mathbf{i}}^{-1} \circ \psi_{\mathbf{i}}(\mathrm{PBW}_{\mathbf{i}}(b)) \\ &= (-x + z - \min\{x, z\}, -y - z + \min\{x, z\}, -\min\{x, z\}) \end{aligned}$$

Applications

Combinatorial side (Minuscule crystals)

When it is of finite classical type, $t =$ minuscule index and $b \in B(\infty)$ such that $b^* \in B(\Lambda_t)$,

we give an explicit description of $\mathcal{D}(b)$ in terms of $M_{\mathbf{i}}$, $N_{\mathbf{i}}$ and the combinatorics of (shifted) Young diagrams.

Note There is a (shifted) Young diagram realization of the minuscule crystal $B(\Lambda_t) \simeq \{$ (shifted) Young diagrams with certain conditions$\}$

Computational side (Periodicity of η_w - maximal parabolic case)

For $w \in \mathrm{W}$, define $\varrho(w) :=$ the smallest positive integer k s.t.

$$\mathcal{D}_w^k(b) \equiv_{\mathrm{fr}} b \qquad \text{for all } b \in \widetilde{B}(w), \quad \text{(up to a multiple of frozens.)}$$

We set

- $\mathrm{W}_t := \langle s_i \mid i \in I \setminus \{t\} \rangle \subset \mathrm{W}$
- $x_t :=$ the longest one in $\mathrm{W}_t \backslash \mathrm{W}$

(Type A_n) For any $t \in I$, we have

$$\varrho(x_t) = \begin{cases} 1 & \text{if } t = 1, n, \\ 2 & \text{if } n = 3 \text{ and } t = 2, \\ n+1 & \text{if } n \geq 4 \text{ and } t = 2, n-1, \\ \dfrac{2(n+1)}{\gcd(n+1, t)} & \text{if } 2 < t < n-1 \end{cases}$$

Observation (based on computer programs using SageMath)

Let $t \in I$. Then $\varrho(x_t)$ looks finite $\iff$ t is minuscule or cominuscule.

(1) When the Weyl group is of type B_n or C_n:

   (a) If $t = 1$, then $\varrho(x_1) = 2$.

   (b) If $t = n$, then $\varrho(x_t) = \begin{cases} 2 & \text{if } n = 2, \\ 4 & \text{otherwise.} \end{cases}$

(2) When the Weyl group is of type D_n:

   (a) If $t = 1$, then $\varrho(x_1) = 2$.

   (b) If $t \in \{n-1, n\}$, then $\varrho(x_t) = \begin{cases} 2 & \text{if } n = 4, \\ 8 & \text{if } n \text{ is odd,} \\ 4 & \text{otherwise.} \end{cases}$

(3) When the Weyl group is of type E_6 and $t \in \{1, 6\}$, we have $\varrho(x_t) = 6$.

(4) When the Weyl group is of type E_7 and $t = 7$, we have $\varrho(x_t) = 4$.

More observation (parabolic cases in type A_5)

$\varrho(x_J)$ for $J \subset I = \{1,2,3,4,5\}$

where x_J = longest one in $W_J \backslash W$, and $W_J = \langle s_j \mid j \in J \rangle$.

J	$\varrho(x_J)$	J	$\varrho(x_J)$
$\emptyset$	6	$\{1,2\}$	10
$\{1,4\}$	16	$\{2,3\}$	10
$\{2,5\}$	16	$\{3,4\}$	10
$\{4,5\}$	10	$\{1,2,3\}$	7
$\{1,2,4\}$	14	$\{1,2,5\}$	14
$\{1,3,4\}$	9	$\{1,4,5\}$	14
$\{2,3,4\}$	2	$\{2,3,5\}$	9
$\{2,4,5\}$	14	$\{3,4,5\}$	7
$\{1,2,3,4\}$	1	$\{1,2,3,5\}$	6
$\{1,2,4,5\}$	4	$\{1,3,4,5\}$	6
$\{2,3,4,5\}$	1	$\{1,2,3,4,5\}$	1

If J is not in the list, then it looks like $\varrho(x_J) = \infty$.

THANK YOU

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