On Fredholm Pfaffians and Riemann-Hilbert problems

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Integrable systems: regularity, non-commutativity and RMT -
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In collaboration with

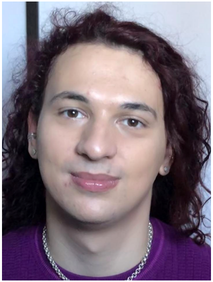

Figure 1: Amari Jaconelli (University of Bristol).

and based on arXiv:2511.23362.

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We want to identify classes of Fredholm Pfaffians which we can analyse asymptotically via Riemann-Hilbert techniques, i.e. via complex analytic methods.

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Using the probabilistic method of Mark Kac (1954), the authors derived asymptotics for certain families of Fredholm Pfaffians.

Those operator Pfaffians were introduced by Rains (2000) and can be viewed as infinite-dimensional Pfaffians, very much like Fredholm determinants provide infinite-dimensional generalisations of determinants. And they are useful!

Begin at the beginning...

If $A = (a_{ij})_{i,j=1}^{2m} = -A^{\intercal}$ is a real $2m \times 2m$ skew-symmetric matrix then

$$\sigma(A) = \{\mathrm{i}\lambda_1, \ldots, \mathrm{i}\lambda_m, -\mathrm{i}\lambda_1, \ldots, -\mathrm{i}\lambda_m\}, \quad \lambda_j \in \mathbb{R},$$

and so $\det(A) = \prod_{j=1}^{m} \lambda_j^2$ is the square of a polynomial in the a_{ij}.

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Pfaff (1815), Jacobi (1827), Cayley (1847)

For an $2m\times 2m$ matrix A, the Pfaffian $\mathrm{Pf}(A)$ of A is defined by

$$\mathrm{Pf}(A):=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\mathrm{sgn}(\sigma)\prod_{j=1}^m a_{\sigma(2j-1)\sigma(2j)}\tag{1}$$

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What transpires, not at all obvious, is that $(\mathrm{Pf}(A))^2 = \det(A)$.

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makes no sense as $m \uparrow \infty$, but

$$\det\left(\mathbb{I} + \begin{bmatrix} 0 & 1/2 & & & \\ 1/2 & 0 & 1/4 & & \\ & 1/4 & & & \\ & & \ddots & & 1/2^{m-1} \\ & & & 1/2^{m-1} & 0 \end{bmatrix}\right)$$

should be fine as $m \uparrow \infty$.

The takeaway message

We should re-express $\det(A) = \det(\mathbb{I} + K)$ in a way that the smallness of K can be exploited as $m \uparrow \infty$.

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Introduce $z \in \mathbb{C}$ and note that

$$\det(\mathbb{I} + zK) = 1 + \sum_{\ell=1}^{m} z^{\ell} \sum_{|\overline{\alpha}|=\ell} \det(K_{ij})_{i,j\in\overline{\alpha}}$$

with $\overline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{N}^\ell$ where $1 \leq \alpha_1 < \ldots < \alpha_\ell \leq m$. This suggests a way of extending the determinant to function spaces.

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Let (X, Σ, μ) be a measure space and $K : X \times X \to \mathbb{C}$ a kernel that defines an integral operator K, acting on $L^2(X, \mathrm{d}\mu)$, through

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Fredholm (1903)

The Fredholm determinant of K on $L^2(X, \mathrm{d}\mu)$ is defined as

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Identity (2) is a numerical identity whenever the RHS is absolutely convergent (which is the case, for instance, if $|K(x,y)| \leq c$ for all $x, y \in X$ and if $\mu(X) < \infty$).

Schatten ideals in $\mathcal{L}(L^2(X, \mathrm{d}\mu))$ - 1940s

If $K : X \times X \to \mathbb{C}$ is continuous and $K \in \mathcal{L}(L^2(X, \mathrm{d}\mu))$ trace class,

$$\|K\|_1 := \sum_{n=1}^{\infty} \int_X \overline{\psi_n(x)} \big(|K|\psi_n\big)(x) \mathrm{d}\mu(x) < \infty$$

where $\{\psi_n\}_{n=1}^{\infty}$ is any ONB of $L^2(X, \mathrm{d}\mu)$, then $z \mapsto d_K(z)$ is entire.

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Trace formula

If $K \in \mathcal{I}_1(L^2(X, \mathrm{d}\mu))$, then $d_K(z) = \prod_{j=1}^{\infty}(1 + z\lambda_j(K))$ and

$$\mathrm{Tr}(K) := \sum_{n=1}^{\infty} \int_X \overline{\psi_n(x)}(K\psi_n)(x)\mathrm{d}\mu(x) = \sum_{j=1}^{\infty} \lambda_j(K).$$

What's more, if $K, M \in \mathcal{I}_1(L^2(X, \mathrm{d}\mu))$ then

$$|d_K(1) - d_M(1)| \leq c\|K - M\|_1, \qquad d_{KM}(1) = d_{MK}(1).$$

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However, if $K \in \mathcal{I}_2(L^2(X, \mathrm{d}\mu))$ is only Hilbert-Schmidt,

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The regularised 2-determinant of $K \in \mathcal{I}_2(L^2(X, \mathrm{d}\mu))$ on $L^2(X, \mathrm{d}\mu)$ is

$$d_{K,2}(z) := \prod_{j=1}^{\infty} \Big(\big(1 + z\lambda_j(K)\big)\mathrm{e}^{-z\lambda_j(K)} \Big). \tag{3}$$

Now do quite the *same* for the Pfaffian

Let K be a $2m \times 2m$ skew-symmetric matrix and $\mathbb{J} := \oplus_{j=1}^{m} \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. We have from (1) that

$$\mathrm{Pf}(\mathbb{J} + zK) = 1 + \sum_{\ell=1}^{m} z^{\ell} \sum_{|\overline{\alpha}|=\ell} \mathrm{Pf}(K_{ij})_{i,j\in\overline{\alpha}},$$

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and which motivates the following $m \uparrow \infty$ extension of the Pfaffian.

Let (X, Σ, μ) be a measure space and $K : X \times X \to \mathbb{C}^{2\times 2}$ a skew-symmetric kernel, i.e.

$$K(x,y) = \begin{bmatrix} K_{11}(x,y) & K_{12}(x,y) \\ -K_{12}(y,x) & K_{22}(x,y) \end{bmatrix}, \quad K_{ii}(x,y) = -K_{ii}(y,x),$$

that defines an integral operator K, acting on $L^2(X, \mathrm{d}\mu) \oplus L^2(X, \mathrm{d}\mu)$, through

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The Fredholm Pfaffian of K on $L^2(X, \mathrm{d}\mu) \oplus L^2(X, \mathrm{d}\mu)$ is defined as

$$p_K(z) := 1 + \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell!} \int_{X^\ell} \mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{\ell} \prod_{m=1}^{\ell} \mathrm{d}\mu(x_m). \tag{4}$$

Identity (4) is a numerical identity whenever the RHS is absolutely convergent.

What's important to notice is that

$$\big(p_K(z)\big)^2 = d_{JK}(-z), \qquad (JK)(x,y) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} K(x,y), \tag{5}$$

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first in the sense of formal power series. But both sides in (5) are absolutely convergent once $K \in \mathcal{I}_1(L^2(X,\mathrm{d}\mu) \oplus L^2(X,\mathrm{d}\mu))$! In the Hilbert-Schmidt class things can be properly regularised.

Ortmann, Quastel, Remenik (2017)

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$$\big(p_{K,2}(z)\big)^2 = d_{JK,2}(-z), \tag{6}$$

for all $z \in \mathbb{C}$.

What do we hope for?

We are aiming to identify classes of skew-symmetric kernels K, involving additional parameters, for which $p_K(z)$, resp. $p_{K,2}(z)$, can be analysed asymptotically.

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(iii) Connection to canonical RHPs.

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(iv) Nonlinear steepest descent analysis, Akhiezer-Kac results.

Setting the stage - derived kernels

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Symplectic and orthogonal derived kernels

The K_{ij} in the definition of K are either of the form

$$K_{12}(y,x) = S(x,y), \quad K_{22}(x,y) = \frac{\partial}{\partial y}S(x,y), \quad K_{11}(x,y) = \int^{x} S(x,y)\mathrm{d}x,$$

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Here, the main kernel $S(x,y)$, its y-derivative and x-antiderivative are assumed to induce trace class operators on $L^2(X)$.

Algebraic simplifications

What results for the Fredholm Pfaffian, resp. determinant?

Bothner-Jaconelli (2025)

In the symplectic derived class, provided $I-2S$ is invertible on $L^2(X)$,

$$d_M(1)=d_{2S}(1)\det\big(\delta_{jk}-F_{jk}(X)\big)_{j,k=1}^{2m},\tag{7}$$

with $F_{jk}(X):=(-1)^k((I-2S^*)^{-1}H)(a_k,a_j), H(x,y)=\int^x S(x,y)\mathrm{d}x$.

A similar result, albeit for the regularised determinant $d_{M,2}(1)$, holds in the orthogonal derived class.

How can we prove (7)?

Operator Pfaffians, resp. determinants, are defined in terms of the spectrum of the underlying integral operator, compare (3), say. Thus one begins by establishing, using commutation identities,

$$\sigma(M) \setminus \{0\} = \sigma(N) \setminus \{0\},$$

where $N \in \mathcal{I}_1(L^2(X) \oplus L^2(X))$ is *simpler*.

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where $N \in \mathcal{I}_1(L^2(X) \oplus L^2(X))$ is *simpler*. Taking multiplicities into account, this leads to an identity of the form

$$d_M(1) = d_{2S^* + \mathrm{rank}(2m)}(1),$$

and from that (7) is derived.

Another approach utilises the aforementioned $d_{AB}(1) = d_{BA}(1)$, i.e. Sylvester's identity, but in that approach the Sobolev space $W^{1,2}(X)$ naturally appears.

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Widom's smoothing out argument

The workings in the orthogonal derived class, because of

$$\epsilon(x) := \frac{1}{2}\begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases},$$

also require a smooth approximation η_n of ϵ.

Connection to RHPs

The least technically involved classes for which (7) can be studied asymptotically are of Hankel composition type,

$$S(x,y)=\frac{1}{2}\bigg(\int_0^{\infty}\phi(x+u)\phi(u+y)\mathrm{d}u-\frac{1}{2}\phi(x)\int_y^{\infty}\phi(z)\mathrm{d}z\bigg),$$

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with $x,y\in X\equiv(t,\infty)$ and of truncated Wiener-Hopf type

$$S(x,y)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\phi(u)\mathrm{e}^{\mathrm{i}u(x-y)}\mathrm{d}u,$$

with $x,y\in X\equiv(-t,t)$. For those, (7) can be further simplified:

Krajenbrink (2021), Bothner-Jaconelli (2025)

If S is of Hankel composition type, then

$$d_M(1) = d_Q(1)\left[1 + \frac{1}{2}\int_t^{\infty}\big((I-Q)^{-1}\phi\big)(x)\Phi(x)\mathrm{d}x\right],\ \Phi(x) := \int_x^{\infty}\phi(z)\mathrm{d}z,\ (8)$$

if $I - 2S, I - Q$ are invertible on $L^2(t,\infty)$. Here, $Q \in \mathcal{L}(L^2(t,\infty))$ is the integral operator with kernel

$$Q(x,y) = \int_0^{\infty}\phi(x+u)\phi(u+y)\mathrm{d}u.$$

This follows from (7) for $m = 1$ with $a_1 = t$ and $a_2 \to \infty$, through an application of the Sherman-Morrison identity.

Bothner-Jaconelli (2025)

If S is of truncated Wiener-Hopf type, then

$$d_M(1) = d_Q(1)\left[1+\frac{1}{2}\int_{-t}^{t}\big((I-Q)^{-1}Q\big)(x,t)\,\mathrm{d}x\right] \times \left[1-\frac{1}{2}\int_{-t}^{t}Q(x,t)\,\mathrm{d}x-\frac{1}{2}\int_{-t}^{t}\left(\int_{-x}^{x}Q(z,t)\,\mathrm{d}z\right)\big((I-Q)^{-1}Q\big)(x,t)\,\mathrm{d}x\right], \tag{9}$$

if $I-2S$ is invertible on $L^2(-t,t)$. Here, $Q\in\mathcal{L}(L^2(-t,t))$ is the integral operator with kernel $Q(x,y):=2S(x,y)$.

This follows from (7) for $m=1$ with $a_1=-t$ and $a_2=t$, using algebraic manipulations.

Both, (8) and (9) have the factor $d_Q(1)$ in common, and that one, for the Hankel composition, resp. truncated Wiener-Hopf, kernel $Q(x, y)$ can be accessed through a canonical RHP.

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Hankel composition RHP

Let $t \in \mathbb{R}$ and $\phi \in W^{1,1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$. Find $X(z) = X(z; t, \phi) \in \mathbb{C}^{2\times 2}$ so

(i) $z \mapsto X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

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(i) $z \mapsto X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(ii) $z \mapsto X(z)$ admits continuous pointwise limits $X_{\pm}(z), z \in \mathbb{R}$ with

$$X_+(z) = X_-(z) \begin{bmatrix} 1 - |r(z)| & -\bar{r}(z)\mathrm{e}^{-\mathrm{i}tz} \\ r(z)\mathrm{e}^{\mathrm{i}tz} & 1 \end{bmatrix}, \ r(z) = -\mathrm{i}\int_{-\infty}^{\infty} \phi(y)\mathrm{e}^{-\mathrm{i}zy}\mathrm{d}y.$$

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(iii) As $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$,

$$X(z) = \mathbb{I} + \frac{1}{z}X_1 + o(z^{-1}), \quad X_1 = \left[X_1^{jk}(t,\phi)\right]_{j,k=1}^2.$$

Bothner (2023)

Provided $I-Q$ is invertible on $L^2(t,\infty)$, the above RHP is uniquely solvable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln d_Q(1)=\mathrm{i}X_1^{11}(t,\phi),\qquad X_1^{21}(t,\phi)=\big((I-Q)^{-1}\phi\big)(t). \tag{10}$$

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What's more

$$1+\frac{1}{2}\int_t^{\infty}\big((I-Q)^{-1}\phi\big)(x)\Phi(x)\mathrm{d}x=\cosh^2\left(\frac{\omega(t)}{2}\right),$$

with

$$\omega(t)=\int_t^{\infty}\big((I-Q)^{-1}\phi\big)(s)\mathrm{d}s.$$

Wiener-Hopf RHP

Let $t > 0$ and $\phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ even with $x\phi(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
Find $X(z) = X(z; t, \phi) \in \mathbb{C}^{2\times 2}$ so

(i) $z \mapsto X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(ii) $z \mapsto X(z)$ admits continuous pointwise limits $X_\pm(z), z \in \mathbb{R}$ with

$$X_+(z) = X_-(z) \begin{bmatrix} 1-\phi(z) & \phi(z)\mathrm{e}^{2\mathrm{i}tz} \\ -\phi(z)\mathrm{e}^{-2\mathrm{i}tz} & 1+\phi(z) \end{bmatrix}.$$

(iii) As $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$,

$$X(z) = \mathbb{I} + \frac{1}{z}X_1 + o(z^{-1}), \quad X_1 = \left[X_1^{jk}(t,\phi)\right]_{j,k=1}^2.$$

Bothner-Jaconelli (2025)

Provided $I-Q$ is invertible on $L^2(-t,t)$, the above RHP is uniquely solvable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln d_Q(1)=-2\mathrm{i}X_1^{11}(t,\phi),\qquad X_1^{21}(t,\phi)=-\mathrm{i}\big((I-Q)^{-1}Q\big)(-t,t).\tag{11}$$

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What's more

$$\text{second factor in (9)}=\cosh^2\left(\frac{\omega(t)}{2}\right),$$

with

$$\omega(t)=2\int_0^t\big((I-Q)^{-1}Q\big)(-s,s)\mathrm{d}s.$$

Nonlinear steepest descent analysis

Equipped with (10) and (11) one analyses the corresponding RHPs as $t \to -\infty$ (Hankel) and $t \to +\infty$ (Wiener-Hopf).

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$$\begin{bmatrix} 1-|r(z)| & -\bar{r}(z)\mathrm{e}^{-\mathrm{i}tz} \\ r(z)\mathrm{e}^{\mathrm{i}tz} & 1 \end{bmatrix} = L(z;t)D(z)U(z;t), \quad z \in \mathbb{R},$$

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where $z \mapsto L(z;t)$, resp. $z \mapsto U(z;t)$, extend to the lower, resp. upper, half-plane and both approach $\mathbb{I}$ fast, as $t \to -\infty$. What results is

Akhiezer-Kac results I

Bothner-Jaconelli (2025)

If S is of Hankel composition type, then as $t \to -\infty$,

$$\ln d_M(1) \sim ts(0) + \int_0^\infty xs(x)s(-x)\mathrm{d}x - \frac{1}{2\pi}\int_{-\infty}^{\infty} \Im\left\{\frac{r'(\lambda)}{r(\lambda)}\right\}\ln(1-|r(\lambda)|^2)\mathrm{d}\lambda$$

$$+ 2\ln\left(\frac{1}{2}\sqrt[4]{\frac{1+\mathrm{i}r(0)}{1-\mathrm{i}r(0)}} + \frac{1}{2}\sqrt[4]{\frac{1-\mathrm{i}r(0)}{1+\mathrm{i}r(0)}}\right), \quad s(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln(1-|r(\lambda)|^2)\mathrm{e}^{\mathrm{i}x\lambda}\mathrm{d}\lambda,$$

if $\|Q\| < 1$ on $L^2(t,\infty)$ and if $\phi \in W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ has

$$|\phi(x)| \leq \mathrm{e}^{-a|x|}, \quad |\phi'(x)| \leq \mathrm{e}^{-a|x|} \quad \forall\, x \in \mathbb{R}$$

with $a \geq 2 + \epsilon$ for some $\epsilon > 0$.

Akhiezer-Kac results II

Bothner-Jaconelli (2025)

If S is of truncated Wiener-Hopf type, then as $t \to +\infty$,

$$\ln d_M(1) \sim -ts(0) + \frac{1}{4}\int_0^\infty xs(x)s(-x)\mathrm{d}x$$
$$+ 2\ln\left(\frac{1}{2}\sqrt[4]{1-\phi(0)} + \frac{1}{2\sqrt[4]{1-\phi(0)}}\right), \quad s(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\ln(1-\phi(\lambda))\mathrm{e}^{\mathrm{i}x\lambda}\mathrm{d}\lambda,$$

if $\|Q\| < 1$ on $L^2(-t,t)$, and if $\phi, x\phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, with ϕ even so $\|\phi\|_\infty < 1$, has

$$z \mapsto \phi(z) \text{ is analytic in some neighbourhood U of } \mathbb{R} \text{ so that } \lim_{\substack{|z|\to\infty \\ z\in U}} \phi(z) = 0.$$

...and go on till you come to the end: then stop.

Various generalisations come to mind, but rather than those what about

(i) Fredholm Pfaffians or determinants acting on $L^2(X, \mathrm{d}\mu)$ with X a higher-dimensional NLS?

Only one thing remains

Thank you very much for your attention!

References I

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