On the Schur elements and generalized graded cellular bases for cyclotomic Hecke-Clifford algebras

Lei Shi

Max-Planck-Institut für Mathematik

Joint work with Shuo Li

18.11.2025

1. Cyclotomic Hecke-Clifford algebras (cHCAs)

2. (Super)symmetrizing forms and Schur elements of cHCAs

3. Generalized graded cellular bases of cHCAs

Hecke-Clifford algebra

Let $\mathbb{K}$ be an algebraically closed field with $\mathrm{Char}(\mathbb{K}) \neq 2$, $q \in \mathbb{K}^*$ satisfying $q^2 \neq \pm 1$, and denote $\epsilon := q - q^{-1} \in \mathbb{K}^*$.

Definition (Hecke-Clifford algebra)

The superalgebra $\mathcal{H}_n^{\mathrm{fin}}$ has the presentation with (even) generators $T_1, \ldots, T_{n-1}$ and (odd) generators $C_1, \ldots, C_n$, subjecting to the relations:

$$T_i^2 = \epsilon T_i + 1, \quad T_i T_j = T_j T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad |i-j| > 1,$$
$$C_i^2 = 1, C_i C_j = -C_j C_i, \quad 1 \leq i \neq j \leq n,$$
$$T_i C_i = C_{i+1} T_i, T_i C_j = C_j T_i, \quad j \neq i, i+1.$$

G. I. Olshanski, Quantized universal enveloping superalgebra of type Q and a super-extension of the Hecke algebra, *Lett. Math. Phys.*, **24** (1992), 93–102.

Affine Hecke-Clifford algebra

Definition (Affine Hecke-Clifford algebra)

The superalgebra $\mathcal{H}_n$ has the presentation with (even) generators $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, $T_1, \ldots, T_{n-1}$ and (odd) generators $C_1, \ldots, C_n$, subjecting to:

$$T_i^2 = \epsilon T_i + 1, \quad T_i T_j = T_j T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad |i-j| > 1,$$
$$C_i^2 = 1, C_i C_j = -C_j C_i, \quad 1 \leq i \neq j \leq n,$$
$$T_i C_i = C_{i+1} T_i, T_i C_j = C_j T_i, \quad j \neq i, i+1.$$

A. Jones and M. Nazarov, Affine Sergeev algebra and q-analogues of the Young symmetrizers for projective representations of the symmetric group, *Proc. London Math. Soc.*, **78** (1999), 481–512.

Affine Hecke-Clifford algebra (continue)

Definition (Continue)

and more relations:

$$
\begin{aligned}
X_iX_j &= X_jX_i, X_iX_i^{-1} = X_i^{-1}X_i = 1 \quad 1 \le i, j \le n,\\
T_iX_i &= X_{i+1}T_i - \epsilon(X_{i+1} + C_iC_{i+1}X_i),\\
T_iX_{i+1} &= X_iT_i + \epsilon(1 + C_iC_{i+1})X_{i+1},\\
T_iX_j &= X_jT_i, \quad j \neq i, i+1,\\
X_iC_i &= C_iX_i^{-1}, X_iC_j = C_jX_i, \quad 1 \le i \neq j \le n.
\end{aligned}
$$

A. Jones and M. Nazarov, Affine Sergeev algebra and q-analogues of the Young symmetrizers for projective representations of the symmetric group, *Proc. London Math. Soc.*, **78** (1999), 481–512.

q-values and b-values

Definition

For any $\iota \in \mathbb{K}^*$, we define

$$\mathtt{q}(\iota) := 2\frac{q\iota + (q\iota)^{-1}}{q + q^{-1}} \in \mathbb{K},$$

$$\mathtt{b}_{\pm}(\iota) := \frac{\mathtt{q}(\iota)}{2} \pm \sqrt{\frac{\mathtt{q}(\iota)^2}{4} - 1} \in \mathbb{K}^*.$$

In fact, we have

$$\mathtt{b}_+(\iota) + \mathtt{b}_-(\iota) = \mathtt{q}(\iota), \quad \mathtt{b}_+(\iota)\mathtt{b}_-(\iota) = 1.$$

Cyclotomic Hecke-Clifford algebra

- We fix $m \geq 0$ and $\underline{Q} = (Q_1, Q_2, \ldots, Q_m) \in (\mathbb{K}^*)^m$.

Definition (Cyclotomic Hecke-Clifford algebra)

We define

$$f := \begin{cases} f^{(0)}_{\underline{Q}} = \prod_{i=1}^{m} \left(X_1 + X_1^{-1} - \mathsf{q}(Q_i) \right), \\ f^{(\mathsf{s})}_{\underline{Q}} = (X_1 - 1) \prod_{i=1}^{m} \left(X_1 + X_1^{-1} - \mathsf{q}(Q_i) \right), \\ f^{(\mathsf{ss})}_{\underline{Q}} = (X_1 - 1)(X_1 + 1) \prod_{i=1}^{m} \left(X_1 + X_1^{-1} - \mathsf{q}(Q_i) \right). \end{cases}$$

Then the cyclotomic Hecke-Clifford algebra $\mathcal{H}^f_n$ is defined as $\mathcal{H}^f_n := \mathcal{H}_n / \mathcal{H}_n f \mathcal{H}_n$.

J. Brundan, A. Kleshchev, Hecke-Clifford superalgebras, crystals of type $A^{(2)}_{2\ell}$ and modular branching rules for $\widetilde{S_n}$, *Represent. Theory,* **5** (2001), 317–403.

$A_{2\ell}^{(2)}$-Categorification

- The field $\mathbb{K}$ is algebraically closed and Char$(\mathbb{K}) \neq 2\ell+1, 2$.
- $\mathfrak{g} = A_{2\ell}^{(2)}$, $I = \{0, 1, \ldots, \ell\}$, $\Lambda \in P_+$ be any dominant integral weight.
- q^2 is a primitive $2l+1$-th root of unity.
- $K(\infty) := \bigoplus_{n\geq 0} K(\text{Rep}\mathcal{H}_n)$.
- $K(\Lambda) := \bigoplus_{n\geq 0} K(\text{Rep}\mathcal{H}_n^{\Lambda})$, where we set

$$f = f_\Lambda = (X_1 - 1)^{a_0} \prod_{i \in I \setminus \{0\}} \left(X_1 + X_1^{-1} - \mathtt{q}(q^{2i})\right)^{a_i},$$

  if $\Lambda = a_0\Lambda_0 + \Sigma_{i\in I\setminus\{0\}} a_i\Lambda_i$.
- $K(\infty)^*$, $K(\Lambda)^*$ are the grading dual of $K(\infty)$, $K(\Lambda)$, respectively.

$A_{2\ell}^{(2)}$-Categorification

Theorem (Brundan-Kleshchev)

(A) There exists isomorphism $K(\infty)^* \simeq U_{\mathbb{Z}}^{+}(\mathfrak{g})$ *as graded (cocommutative) Hopf algebras;*
(B) For any dominant integral weight Λ, *there exists* $U_{\mathbb{Z}}(\mathfrak{g})$*-module isomorphism* $K(\Lambda)^* \simeq V_{\mathbb{Z}}(\Lambda)$;
(C) $\{$*The isomorphic classes of simple modules*$\}\simeq\{$*Kashiwara's crystal basis*$\}$.

J. Brundan, A. Kleshchev, Hecke-Clifford superalgebras, crystals of type $A_{2\ell}^{(2)}$ and modular branching rules for $\widetilde{S_n}$, *Represent. Theory*, **5** (2001), 317–403.

$\mathbb{Z}$-graded versions

- Kang-Kashiwara-Tsuchioka introduced the (cyclotomic) quiver Hecke superalgebras and (cyclotomic) quiver Hecke-Clifford superalgebras, and proved that

Theorem (Kang-Kashiwara-Tsuchioka)

For type $A_\infty, B_\infty, C_\infty, A^{(1)}_{l-1}, A^{(2)}_{2l}, C^{(1)}_{l}, D^{(2)}_{l+1}$, cyclotomic quiver Hecke-Clifford superalgebra RC^Λ_n is isomorphic to cyclotomic Hecke-Clifford superalgebra $\mathcal{H}^\Lambda_n$.

S.-J. Kang, M. Kashiwara and S. Tsuchioka, Quiver Hecke superalgebras, *J. Reine Angew. Math.*, **711** (2016), 1–54.

For $m \geq 0$, we set

$$\mathscr{P}_n^{0,m} := \mathscr{P}_n^m,$$
$$\mathscr{P}_n^{\mathsf{s},m} := \cup_{a=0}^{n} (\mathscr{P}_a^{\mathsf{s}} \times \mathscr{P}_{n-a}^m),$$
$$\mathscr{P}_n^{\mathsf{ss},m} := \cup_{a+b+c=n} (\mathscr{P}_a^{\mathsf{s}} \times \mathscr{P}_b^{\mathsf{s}} \times \mathscr{P}_c^m).$$

We also have the set of standard tableaux $\mathrm{Std}(\mathscr{P}_n^{\bullet,m})$ for $\bullet \in \{0, \mathsf{s}, \mathsf{ss}\}$.

Semisimple representation theory of cHCAs

- Shi-Wan introduced a Poincaré polynomial $P_n^{(\bullet)}(q^2, \underline{Q})$ for the cyclotomic Hecke-Clifford algebras.

Theorem (Shi-Wan, 2025)

Suppose that $P_n^{(\bullet)}(q^2, \underline{Q}) \neq 0$ *for* $\bullet \in \{0, \mathtt{s}, \mathtt{ss}\}$. *Then* $\mathcal{H}_n^f$ *is split semisimple over* $\mathbb{K}$, *and all the simple modules* $\{\mathbb{D}(\underline{\lambda}) \mid \underline{\lambda} \in \mathrm{Std}(\mathscr{P}_n^{\bullet,m})\}$ *are classified.*

L. Shi, J. Wan, *On representation theory of cyclotomic Hecke-Clifford algebras*, arXiv:2501.06763.

Primitive idempotents and seminormal bases of cHCAs

- Based on Shi-Wan's work, Li-Shi constructed the complete set of primitive idempotents and seminormal bases for the (semisimple) cyclotomic Hecke-Clifford algebras (and cyclotomic Sergeev algebras).
- Kashuba-Molev-Serganova constructed the complete set of primitive idempotents (inductively) and seminormal bases for the Sergeev algebra .

S. Li, L. Shi, *Seminormal bases of cyclotomic Hecke-Clifford algebras*, Lett. Math. Phys., **115**, (2025), 10.1007/s11005-025-01998-x.

I. Kashuba, A. Molev and V. Serganova, *On the Jucys-Murphy method and fusion procedure for the Sergeev superalgebra*, J. Lond. Math. Soc. (2), **112**(3) (2025), Paper No. e70302.

Symmetrizing forms of superalgebras

- Let $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ be an R-superalgebra, which is finitely generated free as R-module, $|\cdot| : \mathcal{A} \to \mathbb{Z}_2$ be the parity map.

Symmetrizing forms of superalgebras

- Let $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ be an R-superalgebra, which is finitely generated free as R-module, $|\cdot| : \mathcal{A} \to \mathbb{Z}_2$ be the parity map.

Definition

The superalgebra $\mathcal{A}$ is **symmetric** if $\exists$ R-linear map $t : \mathcal{A} \to \mathrm{R}$ with $t(\mathcal{A}_{\bar{1}}) = 0$ such that $t(xy) = t(yx)$ for any $x, y \in \mathcal{A}$ and t is non-degenerate.

Definition

The superalgebra $\mathcal{A}$ is **supersymmetric** if $\exists$ R-linear map $t : \mathcal{A} \to \mathrm{R}$ with $t(\mathcal{A}_{\bar{1}}) = 0$ such that $t(xy) = (-1)^{|x||y|} t(yx)$ for any homogeneous $x, y \in \mathcal{A}$ and t is non-degenerate.

Frobenius forms of cHCAs

Proposition (Brundan-Kleshchev, 2001)

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in [0, r-1]^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$ *and* $w \in \mathfrak{S}_n$, *we define the* R*-linear map* $\tau_{r,n}$ *by*

$$\tau_{r,n}(X^\alpha C^\beta T_w) := \delta_{(\alpha,\beta,w),(0,0,1)}.$$

Then $\tau_{r,n}$ *is a Frobenius form of* $\mathcal{H}_n^f$ *over* R.

Frobenius forms of cHCAs

Proposition (Brundan-Kleshchev, 2001)

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, r-1]^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_2^n$ *and* $w \in \mathfrak{S}_n$, *we define the* R*-linear map* $\tau_{r,n}$ *by*

$$\tau_{r,n}(X^\alpha C^\beta T_w) := \delta_{(\alpha,\beta,w),(0,0,1)}.$$

Then $\tau_{r,n}$ *is a Frobenius form of* $\mathcal{H}_n^f$ *over* R.

Remark

In general, the form $\tau_{r,n}$ is not (super)symmetric ($\tau_{r,n}(xy) \neq \tau_{r,n}(yx)$)!

(Super)symmetrizing form of cHCAs

Recall the level $r = \deg f$, and $r = 2m$ if $\bullet = \mathsf{0}$, and $r = 2m + 1$ if $\bullet = \mathsf{s}$.

(Super)symmetrizing form of cHCAs

Recall the level $r = \deg f$, and $r = 2m$ if $\bullet = 0$, and $r = 2m+1$ if $\bullet = \mathtt{s}$.

Theorem (Li-Shi, 2025)

Let R *be an integral domain with* $\mathrm{Char}(\mathrm{R}) \neq 2$. *Suppose that* $q, q + q^{-1} \in \mathrm{R}^\times$ *and* $\underline{Q} \in (\mathrm{R}^\times)^m$. *Then we have the following.*

(i) If $\bullet = 0$, *then* $\mathcal{H}_n^f$ *is supersymmetric with the supersymmetrizing form*

$$t_{r,n} := \tau_{r,n}\Big(- \cdot (X_1 X_2 \cdots X_n)^m\Big);$$

(ii) If $\bullet = \mathtt{s}$, *and* $(1+X_1)(1+X_2)\cdots(1+X_n)$ *is invertible in* $\mathcal{H}_n^f$, *then* $\mathcal{H}_n^f$ *is symmetric with the symmetrizing form*

$$t_{r,n} := \tau_{r,n}\Big(- \cdot (X_1 X_2 \cdots X_n)^m (1+X_1)(1+X_2)\cdots(1+X_n)\Big).$$

Symmetrizing form of $\mathcal{H}_n^{\mathrm{fin}}$

Proposition

Suppose that q is not a primitive $4l$-th root of unity for $l \in \{1, 2, \cdots, n\}$, then $\mathcal{H}_n^{\mathit{fin}}$ is a symmetric superalgebra with symmetrizing form $t_{1,n}$.

Remark

Wan and Wang also gave a symmetrizing form $\tau_n^{\mathbf{WW}}$ on $\mathcal{H}_n^{\mathrm{fin}}$ over $\mathbb{C}(q)$, where q is an indeterminate, and computed the corresponding Schur elements.

Schur elements in symmetric case

For any $V \in \mathrm{Irr}(\mathcal{A})$, we write

$$\delta(V) := \begin{cases} 0, & \text{if } V \text{ is of } \textit{type } \mathtt{M}; \\ 1, & \text{if } V \text{ is of } \textit{type } \mathtt{Q}. \end{cases}$$

Schur elements in symmetric case

For any $V \in \mathrm{Irr}(\mathcal{A})$, we write

$$\delta(V) := \begin{cases} 0, & \text{if } V \text{ is of } \textit{type } \mathtt{M}; \\ 1, & \text{if } V \text{ is of } \textit{type } \mathtt{Q}. \end{cases}$$

Proposition (Wan-Wang, 2013)

Suppose $\mathcal{A}$ is a split semisimple superalgebra over $\mathbb{K}$ and $\mathcal{A}$ is symmetric with a symmetrizing form t. Then the **Schur element** c_V *for every irreducible $\mathcal{A}$-module V is non-zero. Moreover,*

$$t = \sum_{V \in \mathrm{Irr}(\mathcal{A})} \frac{1}{2^{\delta_V} c_V} \chi_V.$$

Schur elements in supersymmetric case

Proposition (Li-Shi,2025)

Suppose $\mathcal{A}$ is a split semisimple superalgebra over $\mathbb{K}$ and $\mathcal{A}$ is supersymmetric with a supersymmetrizing form t. Then

Schur elements in supersymmetric case

Proposition (Li-Shi,2025)

Suppose $\mathcal{A}$ is a split semisimple superalgebra over $\mathbb{K}$ and $\mathcal{A}$ is supersymmetric with a supersymmetrizing form t. Then

(i) Every irreducible $\mathcal{A}$-module V is of type $\mathtt{M}$, *and the* **Schur element** c_V *is non-zero.*

Schur elements in supersymmetric case

Proposition (Li-Shi,2025)

Suppose $\mathcal{A}$ is a split semisimple superalgebra over $\mathbb{K}$ and $\mathcal{A}$ is supersymmetric with a supersymmetrizing form t. Then

(i) Every irreducible $\mathcal{A}$-module V is of type $\mathtt{M}$, *and the* **Schur element** c_V *is non-zero.*

(ii) If we fix the choice of representative in each isomorphism class of simple module, then we have

$$t = \sum_{V \in \mathrm{Irr}(\mathcal{A})} \frac{1}{c_V} \chi'_V.$$

Schur elements of cHCAs

Theorem (Li-Shi, 2025)

Suppose that $\bullet \in \{0, \mathsf{s}\}$ *and* $P_n^{(\bullet)}(q^2, \underline{Q}) \neq 0$ *holds. Let* $\underline{\lambda} \in \mathscr{P}_n^{\bullet,m}$*, then the Schur element* $c_{\underline{\lambda}}$ *of simple* $\mathcal{H}_n^f$*-module* $\mathbb{D}(\underline{\lambda})$ *with respect to* $t_{r,n}$ *is given by*

$$c_{\underline{\lambda}} = \begin{cases} \prod_{\alpha \in \underline{\lambda}} (\mathsf{b}_-(\mathrm{res}(\alpha)) - \mathsf{b}_+(\mathrm{res}(\alpha))) \cdot \mathsf{q}(\underline{\lambda})^{-1}, & \text{if } \bullet = 0, \\ 2^{-\lceil \sharp \mathcal{D}_{\underline{\lambda}}/2 \rceil} \cdot \mathsf{q}(\underline{\lambda})^{-1}, & \text{if } \bullet = \mathsf{s}, \end{cases}$$

where

$$\mathsf{q}(\underline{\lambda}) := \prod_{k=1}^{n} \frac{\prod_{\beta \in \mathrm{Rem}(\mathfrak{t}\downarrow_{k-1}) \setminus \mathcal{D}} (\mathsf{q}(\mathrm{res}_{\mathfrak{t}}(k)) - \mathsf{q}(\mathrm{res}(\beta)))}{\prod_{\alpha \in \mathrm{Add}(\mathfrak{t}\downarrow_{k-1}) \setminus \{\mathfrak{t}^{-1}(k)\}} (\mathsf{q}(\mathrm{res}_{\mathfrak{t}}(k)) - \mathsf{q}(\mathrm{res}(\alpha)))},$$

for $\mathfrak{t} \in \mathrm{Std}(\underline{\lambda}), \underline{\lambda} \in \mathscr{P}_n^{\bullet,m}$.

Generalized graded cellular algebra

Theorem (Li-Shi,2025)

Suppose $f = f_{\underline{Q}}^{(0)}$. *The cyclotomic Hecke-Clifford algebra* $\mathcal{H}_{\mathbb{K}}^{\underline{Q}}$ *is a generalized graded cellular superalgebra.*

Definition

Suppose A is a $\mathbb{Z}$-graded R-superalgebra which is free of finite rank over R and R is concentrated on both $\mathbb{Z}$ and $\mathbb{Z}_2$ degree 0. A **generalized graded super cell datum** for A is an ordered hexaple $(\mathscr{P}, \mathscr{T}, \mathscr{B}, C, \deg, |\cdot|)$, where $(\mathscr{P}, \lhd)$ is the **weight poset**, $\mathscr{T}(\lambda)$ and $\mathscr{B}_\lambda$ are finite sets for $\lambda \in \mathscr{P}$, $\mathscr{B}_\lambda$ is a homogeneous R-basis of R-superalgebra B_λ(which is concentrated on $\mathbb{Z}$-degree 0) for $\lambda \in \mathscr{P}$ and $C : \bigsqcup_{\lambda \in \mathscr{P}} \mathscr{T}(\lambda) \times B_\lambda \times \mathscr{T}(\lambda) \to A; (i, u, j) \mapsto c^{\lambda}_{i,u,j}$, $\deg : \bigsqcup_{\lambda \in \mathscr{P}} \mathscr{T}(\lambda) \to \mathbb{Z}$, $|\cdot| : \bigsqcup_{\lambda \in \mathscr{P}} \mathscr{T}(\lambda) \to \mathbb{Z}_2$ are three fuctions such that C is injective. Moreover, we have the following conditions.

Generalized graded cellular algebra

Definition (Continue)

(GCd) Each basis element $c^{\lambda}_{i,u,j}$ is homogeneous of $\mathbb{Z}$-degree $\deg(i)+\deg(j)$ and $\mathbb{Z}_2$-degree $|i|+|j|+|u|$, where $i,j\in\mathscr{T}(\lambda), u\in\mathscr{B}_\lambda, \lambda\in\mathscr{P}$.
(GC1)$\{c^{\lambda}_{i,u,j} \mid i,j\in\mathscr{T}(\lambda), u\in\mathscr{B}_\lambda, \lambda\in\mathscr{P}\}$ forms a homogeneous R-basis of A for $i,j\in\mathscr{T}(\lambda), u\in\mathscr{B}_\lambda, \lambda\in\mathscr{P}$.
(GC2) We have $c^{\lambda}_{i,u,j}+c^{\lambda}_{i,u',j}=c^{\lambda}_{i,u+u',j}$ for $i,j\in\mathscr{T}(\lambda), u,u'\in B_\lambda, \lambda\in\mathscr{P}$.
(GC3) For any $i,j,i',j'\in\mathscr{T}(\lambda), u,u',u''\in\mathscr{B}_\lambda, \lambda\in\mathscr{P}$, we have a function $r^{i',u'}_{i,u}: A\to R: a\mapsto r^{i',u'}_{i,u}(a)$ such that for any $a\in A$ and $c^{\lambda}_{i,u,j}$ where $i,j\in\mathscr{T}(\lambda), u\in\mathscr{B}_\lambda, \lambda\in\mathscr{P}$, we have

$$ac^{\lambda}_{i,uu'',j}=\sum_{\substack{i'\in\mathscr{T}(\lambda)\\ u'\in\mathscr{B}_\lambda}} r^{i',u'}_{i,u}(a)c^{\lambda}_{i',u'u'',j} \pmod{A^{\lhd\lambda}},$$

where

$$A^{\lhd\lambda}:=\sum_{\substack{(i,u,j)\in\mathscr{T}(\mu)\times B_\mu\times\mathscr{T}(\mu)\\ \mu\lhd\lambda}} Rc^{\lambda}_{i,u,j}.$$

Generalized graded cellular algebra

Definition (Continue)

(GC4) For each $\lambda \in \mathscr{P}$, there is an R-algebraic anti-involution ω_λ on B_λ and the R-linear map $* : A \to A$ determined by $(c^\lambda_{i,u,j})^* = c^\lambda_{j,\omega_\lambda(u),i}$ where $i, j \in \mathscr{T}(\lambda), u \in \mathscr{B}_\lambda, \lambda \in \mathscr{P}$ is an anti-isomorphism of A.

A **generalized graded cellular superalgebra** is a $\mathbb{Z}$-graded superalgebra which has a generalized graded super cell datum . The basis $\{c^\lambda_{i,u,j} | \ i, j \in \mathscr{T}(\lambda), u \in \mathscr{B}_\lambda, \lambda \in \mathscr{P}\}$ is a **generalized graded super cellular basis** of A.

Remark

(1).Our generalized graded cellular superalgebra is a special case of generalized standardly based algebra (Mori,2014).
(2).If we forget the $\mathbb{Z}_2$ grading, and $B_\lambda = R$, $\omega_\lambda = \mathrm{id}_R$ for all $\lambda \in \mathscr{P}$,then we recover the definition of $\mathbb{Z}$-graded cellular algebra (Hu-Mathas, 2010).
(3).If we further forget the $\mathbb{Z}$-grading, then we recover the original definition of cellular algebra (Graham-Lehrer,1996).

Generalized graded cellular algebra

Corollary

(1).[Hu-Mathas,2010; Evseev-Mathas,2024] Let R^Λ_α *be the cyclotomic quiver Hecke algebra of type* $A^{(1)}_{l-1}$ *or* $C^{(1)}_l$*. Then* R^Λ_α *is a cellular algebra.*

(2). Let R^Λ_α *be the cyclotomic quiver Hecke superalgebra of type* $A^{(2)}_{2l}$*, and* $\Lambda = a_0\Lambda_0 + \Sigma_{i\in I\setminus\{0\}} a_i\Lambda_i$ *with* $a_0 \in 2\mathbb{Z}$*. Then* $R^\Lambda_\alpha \otimes C_{\ell(\alpha)}$ *is a generalized graded cellular superalgebra.*

(3). Let R^Λ_α *be the cyclotomic quiver Hecke superalgebra of type* $D^{(2)}_{l+1}$*, and* $\Lambda = a_0\Lambda_0 + \Sigma_{i\in I\setminus\{0,l\}} a_i\Lambda_i + a_l\Lambda_l$ *with* $a_0, a_s \in 2\mathbb{Z}$*. Then* $R^\Lambda_\alpha \otimes C_{\ell(\alpha)}$ *is a generalized graded cellular superalgebra.*

Thank You!