Fourier and Schur idempotents

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ICMAT

Connections between commutative and noncommutative Harmonic Analysis
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—Joint with Javier Parcet and Mikael de la Salle—

Introduction
Fourier and Schur multipliers

Classical nonsmooth multipliers

Let $\Omega \subset \mathbf{R}^n$ of non-zero measure, the idempotent multiplier T_{χ_Ω} is given by

$$T_{\chi_\Omega}(f)(\xi) = \int_{\mathbf{R}^n} \chi_\Omega(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

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$\star$ Hilbert transforms

$$H_u(f)(x) = \int_{\mathbf{R}^n} \chi_{\{\xi\in\mathbf{R}^n \mid \langle\xi,u\rangle\geq 0\}}(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi \text{ where } u \in S^{n-1}$$

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Theorem (Fefferman, 1971)

Let $B(0,1)$ be the unit ball of $\mathbf{R}^n$ and $n \geq 2$. Then, the Fourier multiplier

$$T_{\chi_{B(0,1)}}(f)(\xi) = \int_{\mathbf{R}^n} \chi_{\{\xi\in\mathbf{R}^n \mid \ |\xi|\leq 1\}}(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

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⋆ If Ω is a C^1-domain and is not flat, then T_{χ_Ω} is also L^p-unbounded for $p \neq 2$. The curvature of the unit ball is colliding with L^p-boundedness.

Schur multipliers

Let $M : \{1, ..., n\} \times \{1, ..., n\} \to \mathbf{C}$ be a bounded function. The Schur multiplier associated to M is a linear operator $S_M : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ defined by

$$S_M(A) = \Big(M(i,j)A(i,j)\Big)_{1 \leq i,j \leq n}$$

for all $A = \Big(A(i,j)\Big)_{1 \leq i,j \leq n} \in M_n(\mathbf{C})$.

Let (X, μ) be a σ-finite measure space, we will denote

$$S_p(X) = \{x \in B(L^2(X, \mu)) \mid ||x||_p = \mathrm{tr}(|x|^p)^{\frac{1}{p}} < \infty\}.$$

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Recall that $S_2(X) \simeq L^2(X \times X, \mu \times \mu)$, with the identification

$$T \to (T_{x,y})_{x,y \in X}$$

where

$$T(\phi)(x) \sim \int_X T_{x,y} \phi(y) d\mu(y).$$

Let $M : X \times X \to \mathbf{C}$ be a bounded function, the Schur multiplier $S_M : S_2(X) \to S_2(X)$ defined by

$$S_M(T) = \Big(M(x,y)T_{x,y}\Big)_{x,y \in X}$$

for $T = (T_{x,y})_{x,y \in X}$.

A central problem:
Which $M : X \times X \to \mathbf{C}$ define a S_p-bounded Schur multipliers?

Some previous results

Theorem (Grothendieck-Haagerup)

If $M : X \times X \to \mathbf{C}$, *the following conditions are equivalent.*

1. S_M *is bounded on* $B(L^2(X,\mu))$
2. *There exist a Hilbert space* $\mathcal{K}$ *and two uniformly bounded families of vectors* $\{u_x\}_{x\in X}$, $\{v_y\}_{y\in X}$ *such that*

$$M(x,y) = \langle u_x, v_y\rangle_{\mathcal{K}} \;\; a.e. \;\; x,y \in X$$

Remark: *The complete boundedness of* S_M *in* $B(L^2(X,\mu))$ *is also equivalent.*

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$\star$ **Arazy's-Krein's-Koplilenko's conjectures** (Potapov-Sukochev, 2011): If

$$M(x,y) = \frac{f(x)-f(y)}{x-y}$$ for $f : \mathbf{R} \to \mathbf{R}$ Lipschitz, then S_M is S_p-bounded

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⋆ **Non-smooth symbols remain much more unexplored.**

Multipliers in group von Neumann algebras

Let (G, μ) be a unimodular group with

$$\lambda : \mathrm{G} \to \mathcal{U}(L_2(\mathrm{G}, \mu)) \quad \text{given by} \quad \big[\lambda(g)\varphi\big](h) = \varphi(g^{-1}h).$$

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Define its group von Neumann algebra as follows

$$\mathcal{L}\mathrm{G} := \overline{\Big\{ f \, = \, \int_{\mathrm{G}} \widehat{f}(g)\lambda(g) \, d\mu(g) : \widehat{f} \in \mathcal{C}_\mathrm{c}(\mathrm{G}) \Big\}}^{\mathrm{SOT}} \subset \mathcal{B}(L_2(\mathrm{G}, \mu)).$$

The Haar trace τ is then determined by

$$\tau(f) = \widehat{f}(e)$$

and the noncommutative L_p-norms are defined by

$$||f|| = \tau(|f|^p)^{\frac{1}{p}}$$

Given $m : \mathrm{G} \to \mathbf{C}$, its **Fourier multiplier** is the map

$$T_m\Big(\int_G \widehat{f}(g)\lambda(g)d\mu(g)\Big) = \int_G m(g)\widehat{f}(g)\lambda(g)d\mu(g).$$

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Remark: If $G = \mathbf{R}^n$, the map

$$\lambda(\eta) \to e^{2\pi i\eta\cdot} \in L^\infty(\mathbf{R}^n)$$

extends to an isomorphism between $VN(\mathbf{R}^n)$ and $L^\infty(\mathbf{R}^n)$. Therefore, given $m : \mathbf{R}^n \to \mathbf{C}$, the Fourier multiplier T_m is defined by

$$T_m(f)(x) = \int_{\mathbf{R}^n} m(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

In the same spirit as for Schur multipliers, we have

A central problem:
Which $m : G \to \mathbf{C}$ define a L_p-bounded Fourier multiplier?

Transference

The relation between Fourier and Schur multipliers plays a key role...

Fourier-Schur transference [Neuwirth/Ricard + Caspers/de la Salle]

Let G be an amenable group and $1 \leq p \leq \infty$. Let $M : \mathrm{G} \times \mathrm{G} \to \mathbf{C}$ and $m : \mathrm{G} \to \mathbf{C}$ such that $M(g,h) = m(gh^{-1})$. Then

$$\big\| S_M : S_p(\mathrm{G}) \to S_p(\mathrm{G}) \big\|_{\mathrm{cb}} = \big\| T_m : L_p(VN(\mathrm{G})) \to L_p(VN(\mathrm{G})) \big\|_{\mathrm{cb}}.$$

What are the properties of Schur and Fourier indempotents?

Nonsmooth theory for Schur multipliers

Let $\Sigma \subset X$ be a nonzero measure set, and S_Σ be the Schur multiplier with symbol $M(x, y) = \chi_\Sigma(x, y)$. When is S_Σ S_p-bounded?

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Fourier idempotents on Lie groups

Let G be a group and $1 < p \neq 2 < \infty$:

- Which smooth domains Ω give $T_{\chi_\Omega} : L_p(VN(\mathrm{G})) \to L_p(VN(\mathrm{G}))$?
- Is there a geometric characterization? A group theoretic one?

This question is of particular interest in simple Lie groups.

Schur idempotents

Local geometry and analytic form

Local boundedness

Definition (Local S_p-boundedness)

Let $\Sigma \subset X$ be a smooth domain and let $(x_0, y_0) \in \partial\Sigma$. We say that S_Σ is locally S_p-bounded at the point (x_0, y_0), if there exist two neighbourhoods $x_0 \in U \subset X$ and $y_0 \in V \subset X$ such that

$$\left\| S_{\Sigma \cap (U \times V)} : S_p(X) \to S_p(X) \right\| < \infty$$

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- $S_{U \times V}$ is always S_p-bounded since it is just the composition of two projections.
- If Σ is compact and S_Σ is locally S_p-bounded at every $(x_0, y_0) \in \partial\Sigma$, then it is S_p-bounded.

The main result

Let $\Sigma \subset \mathbf{R}^n \times \mathbf{R}^n$ be a $\mathcal{C}^1$-domain.
Given $(x, y) \in \partial\Sigma$, set $\mathbf{n}(x, y) = (\mathbf{n}_1(x, y), \mathbf{n}_2(x, y)) \perp \partial\Sigma$ at (x, y).
A point $(x, y) \in \partial\Sigma$ is called **transverse** when both $\mathbf{n}_1, \mathbf{n}_2$ are nonzero.

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Theorem A (Schur idempotents)

Given $1 < p \neq 2 < \infty$, TFAE for any transverse $(x_0, y_0) \in \partial\Sigma$:

1. **Local S_p-boundedness.** The Schur idempotent S_Σ whose symbol equals 1 on Σ and 0 elsewhere is bounded on $S_p(L_2(U), L_2(V))$ for some pair of neighbourhoods U, V of x_0, y_0 in $\mathbf{R}^n$.

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2. **Zero-curvature condition.** There are neighbourhoods U, V of the points x_0, y_0 such that the vectors $\mathbf{n}_2(x_1, y)$, $\mathbf{n}_2(x_2, y)$ are parallel for any pair of points $(x_1, y), (x_2, y) \in \partial\Sigma \cap (U \times V)$.

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3. **Triangular truncation representation.** There are neighbourhoods U, V of x_0, y_0 and $\mathcal{C}^1$-functions $f_1 : U \to \mathbf{R}$ and $f_2 : V \to \mathbf{R}$, such that the domain $\Sigma \cap (U \times V) = \big\{(x,y) \in U \times V : f_1(x) > f_2(y)\big\}$.

The main result

Theorem A (Schur idempotents)

Let X be a smooth manifold, $\Sigma \subset X$ a smooth domain and let $1 < p \neq 2 < \infty$. TFAE

1. **S_p-boundedness.** S_Σ is locally S_p-bounded on (x_0, y_0).
2. **Triangular representation.** $\Sigma = \big\{ f_1(x) > f_2(y) \big\}$ around (x_0, y_0).

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2. **Triangular representation.** $\Sigma = \big\{ f_1(x) > f_2(y) \big\}$ around (x_0, y_0).

$\star$ **If Σ is compact the S_p-boundedness is global.**
$\star$ **If Σ is not compact the local boundedness is required. There are counterexamples for global boundedness.**

Theorem A holds for differentiable manifolds $M \times N$. This is quite remarkable, since Schur multipliers on general manifolds lack to admit a Fourier transform connection.

Zero-curvature for $\mathcal{C}^2$-domains

Let Σ be a $\mathcal{C}^2$-**domain**:

$$\Sigma \cap (U \times V) = \big\{(x,y) : F(x,y) > 0\big\} \text{ for some } F \in \mathcal{C}^2(\mathbf{R}^n \times \mathbf{R}^n).$$

Equivalent curvature condition for $\mathcal{C}^2$-domains

$$\Big\langle d_x d_y F(x,y), u \otimes v \Big\rangle := u^{\mathrm{t}} \cdot \Big(\partial_{x_j}\partial_{y_k} F(x,y)\Big)_{j,k} \cdot v = 0$$

for all $(x,y) \in \partial\Sigma \cap (U \times V)$ & $(u,v) \in \ker d_x F(x,y) \times \ker d_y F(x,y)$.

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Vanishing forms of $d_{xx}F$ or $d_{yy}F$: Not valid since

$$\Sigma_{\mathrm{r}} = \big\{(x,y) : x \in \Omega\big\} \quad \text{and} \quad \Sigma_{\mathrm{c}} = \big\{(x,y) : y \in \Omega\big\}$$

lead to S_p-bounded multipliers with no geometric restrictions on Ω.

Fourier idempotents

Three Hilbert transforms on Lie groups

Fourier idempotents so far...

Euclidean idempotents

Semispaces and (finite or lacunary) combinations of them.

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Free groups

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Crossed products

Parcet-Rogers 2016: Twisted Hilbert transforms $H_u \rtimes \mathrm{id}$ over $\mathbf{R}^n \rtimes \mathrm{G}$
$H_u \rtimes \mathrm{id}$ L_p-bounded if and only if the orbit $\{g \cdot u : g \in \mathrm{G}\}$ is finite

Local boundedness

Definition (Local boundedness)

Let $m : G \to \mathbf{C}$ be a bounded function and let $g_0 \in G$. We say that T_m is locally L_p-bounded (respectively cb) near g_0 if there exists a constant C and neighbourhood $g_0 \in U \subset G$ such that

$$\Big\| T_m\Big(\int_G \widehat{f}(g)\lambda(g)d\mu(g)\Big)\Big\|_p \leq C \Big\| \int_G \widehat{f}(g)\lambda(g)d\mu(g)\Big\|_p$$

for all $\widehat{f}$ such that $\mathrm{supp}(\widehat{f}) \subset U$.

Remark: If a symbol $m : G \to \mathbf{C}$ is compactly supported and locally L_p-bounded at every $g_0 \in G$, then it is L_p-bounded.

Theorem B1 (Fourier idempotents)

Let G be a connected Lie group. Let $\Omega \subset \mathrm{G}$ a $\mathcal{C}^1$-domain and $g_0 \in \partial\Omega$. Then, TFAE for every $1 < p \neq 2 < \infty$:

- T_{χ_Ω} is locally L_p-bounded at $g_0 \in G$.
- There exist a neighbourhood U such that of $g_0 \in G$ such that
$$\Omega \cap U = g_0 \cdot \exp(\mathfrak{h}) \cap U$$
where $\mathfrak{h}$ is a Lie subalgebra of codimension 1.

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This unravels what is "Fourier boundary flatness" for Lie groups

Fourier idempotents – The group structure

Consider:

i) The real line $\mathrm{G}_1 = \mathbf{R}$ with $\Omega_1 = (0, \infty)$.

ii) The affine group $\mathrm{G}_2 = \mathrm{Aff}_+(\mathbf{R})$ and $\Omega_2 = \{ax + b : b > 0\}$.

iii) The universal covering $\mathrm{G}_3 = \widetilde{\mathrm{PSL}}_2(\mathbf{R})$ and $\Omega_3 = \{g : \alpha_g(0) > 0\}$.
$\alpha : \widetilde{\mathrm{PSL}}_2(\mathbf{R}) \curvearrowright \mathbf{R}$ by lifting standard action $\mathrm{PSL}_2(\mathbf{R}) \curvearrowright P1(\mathbf{R})$ to universal covers.

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$\alpha : \widetilde{\mathrm{PSL}}_2(\mathbf{R}) \curvearrowright \mathbf{R}$ by lifting standard action $\mathrm{PSL}_2(\mathbf{R}) \curvearrowright P1(\mathbf{R})$ to universal covers.

Theorem B2 (Fourier idempotents)

Let G be simply connected and Ω, p, g_0 as above. Then, TFAE:

- T_{χ_Ω} defines locally at g_0 a Fourier cb-L_p-multiplier.
- There exists a smooth surjective homomorphism $f : G \to G_j$, and a neighbourhood $g_0 \in U \subset G$ such that

$$\Omega \cap U = g_0 f^{-1}(\Omega_j) \cap U.$$

Fourier idempotents – The group structure

Consider:

i) The real line $\mathrm{G}_1 = \mathbf{R}$ with $\Omega_1 = (0, \infty)$.

ii) The affine group $\mathrm{G}_2 = \mathrm{Aff}_+(\mathbf{R})$ and $\Omega_2 = \{ax + b : b > 0\}$.

iii) The universal covering $\mathrm{G}_3 = \widetilde{\mathrm{PSL}}_2(\mathbf{R})$ and $\Omega_3 = \{g : \alpha_g(0) > 0\}$.
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These three Hilbert transforms are globally L_p-bounded

Corollary B3 (Nilpotent and Simple Lie groups)

Fourier cb-L_p-idempotents for $1 < p \neq 2 < \infty$:

i) **Simply connected nilpotent Lie groups**
They are locally $= H \circ \varphi$ for $\varphi : \mathrm{G} \to \mathbf{R}$ smooth hom.

ii) **Simple Lie groups not locally isomorphic to** $\mathrm{SL}_2(\mathbf{R})$
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Applications for nilpotent and simple Lie groups

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There are no L_p-bounded idempotent multipliers in $SL_n(\mathbf{R})$ for $p \neq 2$ and $n \geq 3$. No L_p-approximations by means of nonsmooth symbols.

Ingredients of the proofs

Schur idempotents

Lemma A1 (Schur amplification of Meyer's lemma)

Assume $\partial\Sigma$ transverse in $U \times V$ and $S_\Sigma \in \mathcal{B}(S_p(L_2(U), L_2(V)))$.
Given $z_j = (x_j, y) \in \partial\Sigma \cap (U \times V)$, $u_j = \mathbf{n}_2(z_j)$ and $f_j \in L_p(\mathbf{R}^n)$:

$$\Big\| \Big(\sum_j \big| H_{u_j}(f_j) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathbf{R}^n)} \lesssim \Big\| \Big(\sum_j |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathbf{R}^n)}.$$

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Lemma A2 (Local normal form of transverse hypersurfaces)

If $(x_0, y_0) \in \partial\Sigma$ is transverse, there are local diffeomorphisms st

$$\phi(x_0) = \psi(y_0) = 0 \quad \text{and} \quad \phi \times \psi(\Sigma) = \Big\{ \big((s, \tilde{x}), y\big) : s > g(\tilde{x}, y) \Big\}$$

for some $g \in \mathcal{C}^1(\mathbf{R}^{n-1} \times \mathbf{R}^n)$ satisfying $g(0, y) = \langle y, e_1 \rangle$ for every y.

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Lemma A3 (Measurable transformations of Schur S_p-multipliers)

Let $(X, \mu), (X', \mu')$ be atomless σ-finite and $f, g : X \to X'$ be measurable. Then, if $f_*\mu << \mu'$ and $m \in L_\infty(X' \times X')$, we obtain
$\|m \circ (f \times g)\|_{MS_p(L_2(X,\mu))} = \|m\|_{MS_p(L_2(X', f_*\mu))} \leq \|m\|_{MS_p(L_2(X',\mu'))}$.

Fourier idempotents

PRdlS'22 – If G unimodular and $p \in 2\mathbf{Z}$
Fourier and Schur multipliers are locally the same.

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The result below generalizes Parcet-Ricard-de la Salle's local transference...

Theorem C (Local Fourier-Schur transference)

Let G be a locally compact group and consider a bounded measurable function $m : \mathrm{G} \to \mathbf{C}$. Then, TFAE for $g_0 \in \mathrm{G}$ and every $1 < p < \infty$:

- There is a neighbourhood U of g_0 such that the restriction of T_m to the space of elements of $L_p(\mathcal{L}\mathrm{G})$ Fourier supported by U is cb.
- There are open sets $V, W \subset \mathrm{G}$ with $g_0 \in VW^{-1}$ such that the function $(g,h) \in V \times W \mapsto m(gh^{-1})$ is in $M_{\mathrm{cb}}S_p(L_2(V), L_2(W))$.

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Lie algebra analysis

Zero-curvature $\rightsquigarrow$ $\partial\Omega = g_0 \exp(\mathfrak{h})$.
Lie's classification of codim-1 Lie subalgebras $\rightsquigarrow$ 3 Hilbert transforms