Riesz-Schur transforms

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Riesz transforms

Commutative and Noncommutative

Euclidean Riesz transforms

Classical Riesz transforms

$$R_j f(x) = \partial_j (-\Delta)^{-\frac{1}{2}} f(x) = \int_{\mathbf{R}^n} \frac{\xi_j}{|\xi|} \widehat{f}(\xi) e^{-2\pi i \langle x, \xi \rangle} d\xi$$

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Dimension free estimates (Gundy-Varopoulos'79 / Stein'83)

Given $1 < p < \infty$, the following holds with dim-free contants

$$\|f\|_{L_p(\mathbf{R}^n)} \approx \big\|\nabla(-\Delta)^{-\frac{1}{2}} f\big\|_{L_p(\mathbf{R}^n;\ell_2(n))} = \Big\|\Big(\sum_{j=1}^n |R_j f|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathbf{R}^n)}.$$

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Let $A_\beta = (-\Delta)^\beta$ for $\beta > 0$ and set $\Gamma_\beta(f_1, f_2) = \frac{1}{2}\Big(\overline{A_\beta(f_1)} f_2 + \overline{f_1} A_\beta(f_2) - A_\beta(\overline{f_1} f_2)\Big)$.

When $\beta = 1$ we get $\Gamma_1(f, f) = \|\nabla f\|_{\ell_2(n)}^2$ and $\Gamma_1\big(A_1^{-\frac{1}{2}} f, A_1^{-\frac{1}{2}} f\big)^{\frac{1}{2}} = \big\|\nabla(-\Delta)^{\frac{1}{2}} f\big\|_{\ell_2(n)}$.

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Failure for Poisson Riesz transforms below L_2 (Fefferman'70)

Let $\beta = \frac{1}{2} \rightsquigarrow \|f\|_{L_p(\mathbf{R}^n)} \approx \Big\| \Gamma_\beta \big(A_\beta^{-\frac{1}{2}} f, A_\beta^{-\frac{1}{2}} f \big)^{\frac{1}{2}} \Big\|_{L_p(\mathbf{R}^n)}$ if and only if $p > \frac{2n}{n+1}$.

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Lust-Piquard (1998-2004): **Other important Riesz transforms** $\rightsquigarrow$ **Naor's work** (2016).

Noncommutative Riesz transforms

How to define the Riesz symbols on other groups?

Topological groups + Hilbert spaces

The **Riesz symbols** $m_j(\xi) = \xi_j/|\xi| = \langle \xi, e_j \rangle / \langle \xi, \xi \rangle^{\frac{1}{2}}$ exploit the underlying Hilbert space structure of Euclidean spaces. The idea to define analogous symbols in other topological groups is to work with **suitable maps** $\beta : \mathrm{G} \to \mathcal{H}$ and define

$$m_j(g) = \frac{\langle \beta(g), e_j \rangle_{\mathcal{H}}}{\sqrt{\langle \beta(g), \beta(g) \rangle_{\mathcal{H}}}} \quad \text{for a given ONB} \quad \{e_j : j \geq 1\} \subset \mathcal{H}.$$

In this point, it is very important to admit **infinite-dimensional Hilbert spaces** $\mathcal{H}$ above.

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Which maps $\beta : \mathrm{G} \to \mathcal{H}$ preserve information on G?

We need an **action** $\alpha : \mathrm{G} \curvearrowright \mathcal{H}$ so that $(\alpha, \beta, \mathcal{H})$ is a **cocycle**: $\alpha_g(\beta(h)) = \beta(gh) - \beta(g)$.

Schoenberg thm (1938): Cocycles $\leftrightarrow$ Affine representations $\leftrightarrow$ Markov semigroup gentrs.

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Noncommutative Riesz transforms for topological groups

If $f = \int_{\mathrm{G}} \widehat{f}(g) \lambda(g) \, d\mu(g) \ \rightsquigarrow \ R_{\beta j} f = \int_{\mathrm{G}} \frac{\langle \beta(g), e_j \rangle_{\mathcal{H}}}{\sqrt{\langle \beta(g), \beta(g) \rangle_{\mathcal{H}}}} \widehat{f}(g) \lambda(g) \, d\mu(g) = \partial_{\beta, j} A_\beta^{-\frac{1}{2}} f.$

Left reg rep $\lambda : \mathrm{G} \to \mathcal{U}(L_2(\mathrm{G}))$ with $f \in \mathcal{L}(\mathrm{G})$ = group vNa of $\mathrm{G} = L_\infty(\widehat{\mathrm{G}})$ for G abelian.

HA in group vNas: **Fundamental work of Haagerup** (1979) + **NC L_p-theory** (2010-2025).

Dimension free estimates and applications

Theorem A – Dimension free estimates (Junge-Mei-Parcet'18)

Given $1 < p < \infty$, we have:

$$\|f\|_p \sim_{c(p)} \begin{cases} \inf\limits_{R_{\beta j} f = a_j + b_j} \Big\| \Big(\sum\limits_{j \geq 1} a_j^* a_j \Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum\limits_{j \geq 1} \widetilde{b}_j \widetilde{b}_j^* \Big)^{\frac{1}{2}} \Big\|_p & p \leq 2, \\ \max \Big\{ \Big\| \Big(\sum\limits_{j \geq 1} |R_{\beta j}(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p, \Big\| \Big(\sum\limits_{j \geq 1} |R_{\beta j}(f^*)|^2 \Big)^{\frac{1}{2}} \Big\|_p \Big\} & p \geq 2. \end{cases}$$

The $\widetilde{b}_j$'s are twisted forms of the b_j's, which coincide with b_j when the action α is trivial.

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$$(\mathrm{G}, \mathcal{H}, \alpha_\xi, \beta(\xi)) = (\mathbf{R}^n, \mathbf{R}^n, \mathrm{Id}, \xi).$$

Fractional laplacians

$$(\mathrm{G}, \mathcal{H}, \alpha_\xi, \beta(\xi)) = \big(\mathbf{R}^n, L_2(\mathbf{R}^n, \mu_\beta), f \mapsto \exp(2\pi i \langle\ , \xi\rangle) f, 1 - \exp(2\pi i \langle\ , \xi\rangle)\big).$$

The key is the crossed product $\mathcal{H} \rtimes_\alpha \mathrm{G}$ $\rightsquigarrow$ **Might not be abelian** $\rightsquigarrow$ **Even for** G **abelian!**

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Key – *HM multipliers = LP averages of Riesz transforms for fractional laplacians!*
This produces new Hörmander-Mikhlin L_p-multipliers in group von Neumann algebras.
I learnt from Éric Ricard that this phenomenon was hidden in Bourgain's vector-valued work.

Schur multipliers

An extension of Fourier multipliers

What are Schur multipliers?

If $M : \{1, 2, \ldots, n\}^2 \to \mathbf{C}$, define for $A \in M_n$

$$S_M(A) := \Big(M(j,k) A_{jk} \Big)_{j,k}.$$

If $M : \mathbf{Z} \times \mathbf{Z} \to \mathbf{C}$, define S_M for infinite matrices $A \in \mathcal{B}(\ell_2(\mathbf{Z}))$.

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If $M : \Omega \times \Omega \to \mathbf{C}$ and $T \in \mathcal{B}(L_2(\Omega, \mu))$ admits a kernel K, define

$$S_M(T)f(x) = \int_\Omega M(x,y)K(x,y)f(y)d\mu(y).$$

No worries: Operators with kernel are dense in the relevant topologies below!

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A far-reaching goal...

Which $M : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ satisfy

$$\big\|S_M(A)\big\|_{S_p(\mathbf{R}^n)} = \operatorname{tr}\Big(\big(S_M(A)^* S_M(A)\big)^{\frac{p}{2}} \Big)^{\frac{1}{p}} \le C_p \|A\|_{S_p(\mathbf{R}^n)}?$$

Or more generally, same the problem with $M\colon \Omega \times \Omega \to \mathbf{C}$ for other measure spaces (Ω,μ).

The Fourier-Schur transference theorem

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Fourier-Schur transference (Neuwirth-Ricard'11 / Caspers-de la Salle'15 / PRdlS'22)

If $1 \leq p \leq \infty$

$$\Big\| S_m : S_p(\mathbf{R}^n) \to S_p(\mathbf{R}^n) \Big\|_{\mathsf{cb}} = \Big\| T_m : L_p(\mathbf{R}^n) \to L_p(\mathbf{R}^n) \Big\|_{\mathsf{cb}}.$$

The same holds for multipliers in **amenable** groups and **locally for nonamenable** groups.

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$$L_\infty(\mathbf{T}) \ni f = \sum_{n \in \mathbf{Z}} \widehat{f}(n) e^{2\pi i n \cdot} \mapsto \Big(\widehat{f}(j-k) \Big)_{j,k} \in \mathcal{B}(\ell_2(\mathbf{Z})).$$

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- ⋆ Pioneering work of Haagerup '79 + coauthors.
- ⋆ L_p**-theory** since 2010: Lafforgue, de Laat, de la Salle, Junge, Mei, Parcet, Ricard, Xu...
- ⋆ **Approx properties** $\approx$ Fourier L_p-summability $\rightsquigarrow$ Geometric group th + Operator algebras

Known results for nonToeplitz Schur multipliers

Grothendieck's characterization'56

S_M bounded on $\mathcal{B}(L_2(\Omega))$ iff S_M cb-bded iff there exists Hilbert space $\mathcal{K}$ and uniformly bded (measurable) families (u_x) and (w_y) in $\mathcal{K}$ st $M(x,y) = \langle u_x, w_y \rangle_{\mathcal{K}}$ for a.e. $x, y \in \Omega$.

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Arazy's and Krein's conjectures (Potapov-Sukochev'11)

Given $f : \mathbf{R} \to \mathbf{C}$ Lipschitz and $1 < p < \infty$

$$M_f(x,y) = \frac{f(x) - f(y)}{x - y} \rightsquigarrow \big\| S_{M_f} : S_p(\mathbf{R}) \to S_p(\mathbf{R}) \big\|_{\mathrm{cb}} \leq C_p \|f\|_{\mathrm{Lip}}.$$

Moreover, if $A, B \in S_p(\mathbf{R})$ self-adjoint $\|f(A) - f(B)\|_{S_p(\mathbf{R})} \leq C_p \|f\|_{\mathrm{Lip}} \|A - B\|_{S_p(\mathbf{R})}$.

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S_M bounded on $\mathcal{B}(L_2(\Omega))$ iff S_M cb-bded iff there exists Hilbert space $\mathcal{K}$ and uniformly bded (measurable) families (u_x) and (w_y) in $\mathcal{K}$ st $M(x,y) = \langle u_x, w_y \rangle_{\mathcal{K}}$ for a.e. $x, y \in \Omega$.

Arazy's and Krein's conjectures (Potapov-Sukochev'11)

Given $f : \mathbf{R} \to \mathbf{C}$ Lipschitz and $1 < p < \infty$

$$M_f(x,y) = \frac{f(x) - f(y)}{x - y} \rightsquigarrow \big\| S_{M_f} : S_p(\mathbf{R}) \to S_p(\mathbf{R}) \big\|_{\mathrm{cb}} \leq C_p \|f\|_{\mathrm{Lip}}.$$

Moreover, if $A, B \in S_p(\mathbf{R})$ self-adjoint $\|f(A) - f(B)\|_{S_p(\mathbf{R})} \leq C_p \|f\|_{\mathrm{Lip}} \|A - B\|_{S_p(\mathbf{R})}$.

Hörmander-Mikhlin-Schur multipliers (Conde-González-Parcet-Tablate'23)

$$\big\| S_M \big\|_{\mathrm{cb}(S_p(\mathbf{R}^n))} \lesssim \sum_{|\gamma| \leq [\frac{n}{2}]+1} \Big\| |x-y|^{|\gamma|} \Big\{ \big| \partial_x^\gamma M(x,y) \big| + \big| \partial_y^\gamma M(x,y) \big| \Big\} \Big\|_\infty$$

The proof = Transference + New interpolation results + Noncommutative CZ methods

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Other important nonToeplitz results

- **Marcinkiewicz's multiplier theorem** (Chuah-Liu-Mei'24)
- **Fefferman's ball multiplier theorem** (Parcet-de la Salle-Tablate'25)

Riesz-Schur transforms

And new inequalities for Schur multipliers

Matrix Riesz transforms

Given $p > 2$ and $(a_j, h_j) \in S_p(\Gamma) \times \mathcal{H}$

$$\Big\| \sum_{j \geq 1} a_j \otimes h_j \Big\|_{RC_p} := \max \left\{ \Big\| \Big(\sum_{j,k \geq 1} \langle h_j, h_k \rangle a_j^* a_k \Big)^{\frac{1}{2}} \Big\|_p, \Big\| \Big(\sum_{j,k \geq 1} \langle \overline{h_j}, \overline{h_k} \rangle a_j a_k^* \Big)^{\frac{1}{2}} \Big\|_p \right\}.$$

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If W is a gaussian functor on $\mathcal{H}$, the NC Khintchine inequalities gives

$$\Big\| \sum_{j\geq 1} a_j \otimes h_j \Big\|_{RC_p} \approx_{c_p} \mathbf{E} \Big\| \sum_{j\geq 1} a_j \otimes \mathrm{W}(h_j) \Big\|_{S_p(\Gamma)}.$$

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Theorem B - Matrix Riesz transforms (Arhancet-Kriegler'22 / Glez-Parcet-Pérez-Ricard'25)

Let $1 < p < \infty$ and $(u_j)_{j\in\Gamma} \subset \mathcal{H}$ distinct vectors. Then

$$\Big\| \Big(a_{jk}(1-\delta_{jk}) \Big) \Big\|_{S_p(\Gamma)} = \Big\| \sum_{j\neq k} a_{jk} e_{jk} \Big\|_{S_p(\Gamma)} \approx_{c_p} \Big\| \sum_{j\neq k} a_{jk} e_{jk} \otimes \frac{u_j - u_k}{\|u_j - u_k\|} \Big\|_{RC_p}.$$

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- ⋆ **Riesz-Schur transforms** $\mathcal{R}_u(A) = \Big(a_{jk} \otimes \frac{u_j - u_k}{\|u_j - u_k\|} \Big)$.
- ⋆ Thm B = **NonToeplitz** extension of Thm A (also ok for σ-finite Ω).
- ⋆ **Euclidean case** (Stein) by Fourier-Schur transference using $\mathcal{H} = \Gamma = \mathbf{R}^n$ and $u_x = x$.
- ⋆ **Easy/Conceptual proof from Grothendieck's ideas** (No CZ / Probability) $\rightsquigarrow$ **Applications**.

Application 1 – Group Riesz transforms

Corollary B1 - Group Riesz transforms (GPPR'25)

Let $2 < p < \infty$ and consider G amenable equipped with a cocycle $(\alpha, \beta, \mathcal{H})$. Then

$$\|f\|_{L_p^\circ(\mathcal{L}(\mathrm{G}))} \approx_{c_p} \Big\| \sum_{g \in \mathrm{G}} \widehat{f}(g)\lambda(g) \otimes \frac{\beta(g)}{\|\beta(g)\|} \Big\|_{C_p} + \Big\| \sum_{g \in \mathrm{G}} \widehat{f}(g)\lambda(g) \otimes \frac{\beta(g^{-1})}{\|\beta(g^{-1})\|} \Big\|_{R_p}.$$

A very clean form of dim-free estimates for Riesz transforms: **The case $p < 2$ by duality**.
It easily follows from Fourier-Schur transference and $\alpha_{g^{-1}}(\beta(gh^{-1})) = \beta(h^{-1}) - \beta(g^{-1})$.

Corollary B2 - Grothendieck's type S_p-criterion (GPPR'25)

Let $\{u_j, u'_j, w_j, w'_j : j \in \Gamma\} \subset \mathcal{H}$. Then, we get for $1 < p < \infty$

$$\big\| S_M : S_p(\Gamma) \to S_p(\Gamma) \big\|_{\mathrm{cb}} \le C_p \quad \text{for} \quad M(j,k) = \left\langle \frac{u_j + u'_k}{\|u_j + u'_k\|}, \frac{w_j + w'_k}{\|w_j + w'_k\|} \right\rangle.$$

Taking $\mathcal{H} = \Gamma = \mathbf{R}$ and $(u_j, u'_k, w_j, w'_k) = (j, -k, 1, 0)$ we get the **triangular truncation**.
Taking $u'_k = w_j = 0$ yields a weaker form of **Grothendieck's criterion** for Schatten p-classes.

Application 3 – Hörmander-Mikhlin-Schur multipliers

Corollary B3 - Hörmander-Mikhlin-Schur multipliers (GPPR'25)

Let $1 < p < \infty$ and $M \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathbf{R}^{2n} \setminus \{0\})$

$$\big\| S_M \big\|_{\mathrm{cb}(S_p(\mathbf{R}^n))} \lesssim \sum_{|\gamma| \le [\frac{n}{2}]+1} \Big\| |x-y|^{|\gamma|} \Big\{ \big| \partial_x^\gamma M(x,y) \big| + \big| \partial_y^\gamma M(x,y) \big| \Big\} \Big\|_\infty .$$

The new proof avoids transference and Calderón-Zygmund methods.
It uses "*HM = LP average of RT_β's*" and yields a stronger Sobolev statement .

Similar ideas allow us to recover Marcinkiewicz-Schur multiplier thm (Chuah-Liu-Mei'24).

Application 4 – Arazy's conjecture and divided differences

Corollary B4 - Improvements of Arazy's conjecture (CGPT'23 + GPPR'25)

Let $1 < p < \infty$ and $f : \mathbf{R} \to \mathbf{R}$

- If f is **Lipschitz nondecreasing**

$$M_{\mathrm{sq},f}(x,y) = \sqrt{\frac{f(x)-f(y)}{x-y}} \rightsquigarrow \big\| S_{M_{\mathrm{sq},f}} : S_p(\mathbf{R}) \to S_p(\mathbf{R}) \big\|_{\mathrm{cb}} \leq C_p \|f\|_{\mathrm{Lip}}^{\frac{1}{2}}.$$

- If f is α-**Hölder** and $\left|\frac{1}{2} - \frac{1}{p}\right| < \min\left\{\alpha, \frac{1}{2}\right\}$

$$M_{\alpha f}(x,y) = \frac{f(x)-f(y)}{|x-y|^{\alpha}} \rightsquigarrow \big\| S_{M_{\alpha f}} : S_p(\mathbf{R}) \to S_p(\mathbf{R}) \big\|_{\mathrm{cb}} \leq C_p \|f\|_{\Lambda_\alpha}.$$

These two results may be combined/interpolated to provide even more general statements.

Thanks!