Representations of Quantum Symmetric Pairs at Roots of 1

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Quantum groups

- $\mathfrak{g} = \mathbb{C}\langle e_i, f_i, h_i \mid i \in I\rangle / \sim$: complex semisimple Lie algebra;
- $P = \mathbb{Z}[\omega_i \mid i \in I]$: weight lattice; $Q = \mathbb{Z}[\alpha_i \mid i \in I]$: root lattice
- $\mathrm{U} = U_q(\mathfrak{g}) = \mathbb{C}(q)\langle E_i, F_i, K_\mu \mid i \in I, \mu \in P\rangle / \sim$: simply-connected quantum group

$$
\begin{aligned}
&K_\mu E_i = q^{\langle \mu, \alpha_i \rangle} E_i K_j, \quad K_\mu F_i = q^{-\langle \mu, \alpha_i \rangle} F_i K_j,\\
&[E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}},\\
&\sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1 - c_{ij} \\ r \end{bmatrix}_i E_i^r E_j E_i^{1-c_{ij}-r} = 0 \quad (i \neq j),\\
&\sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1 - c_{ij} \\ r \end{bmatrix}_i F_i^r F_j F_i^{1-c_{ij}-r} = 0 \quad (i \neq j).
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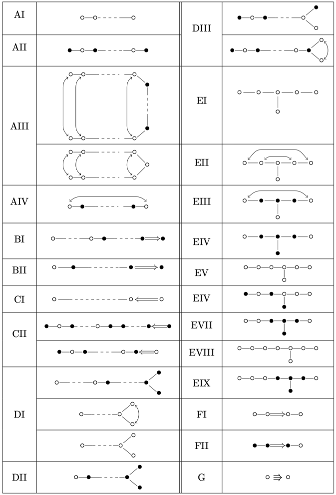
Symmetric pairs and Satake diagrams

- $\mathfrak{g}$: complex semisimple Lie algebra;
- $\theta : \mathfrak{g} \to \mathfrak{g}$: involution on $\mathfrak{g}$;
- $\mathfrak{k} = \mathfrak{g}^{\theta}$: reductive subalgebra
- $(\mathfrak{g}, \mathfrak{k})$: symmetric pair $\rightsquigarrow$
  $(I = I_{\circ} \sqcup I_{\bullet}, \tau)$: Satake diagram

Table 4. Satake diagrams of irreducible symmetric pairs



Quantum symmetric pairs

- $(U(\mathfrak{g}), U(\mathfrak{k}))$: symmetric pair $\rightsquigarrow$ $(\mathrm{U}, \mathrm{U}^\imath)$: quantum symmetric pairs

Definition (Letzter, Kolb)

The $\imath$quantum group $\mathrm{U}^\imath$ associated with $(\mathfrak{g}, \mathfrak{k})$ is the $\mathbb{C}(q)$-subalgebra of U generated by

$$\begin{aligned} &B_i = F_i + \varsigma_i T_{w_\bullet}(E_{\tau i}) K_i^{-1}, \qquad (i \in I_\circ), \\ &E_j, \quad F_j, \qquad (j \in I_\bullet), \\ &K_\mu, \qquad (\mu \in P^\theta) \end{aligned}$$

- coideal subalgebra: $\Delta : \mathrm{U}^\imath \to \mathrm{U}^\imath \otimes \mathrm{U}$
- For the pair $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_\Delta)$, one has $\mathrm{U}^\imath \cong \mathrm{U}$.

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Answer: Poisson geometry of $\mathrm{U}^\imath$!

Integral forms

Let $\mathcal{A} = \mathbb{C}[q, q^{-1}]$. There are three distinguished integral forms:

$$\mathrm{U}_{\mathcal{A}}^{Lu} = \mathcal{A}\langle E_i^{(n)} = E_i^n/[n]_i!, F_i^{(n)}, K_\mu\rangle : \text{Lusztig form}$$

$$\mathrm{U}_{\mathcal{A}}^{DK} = \mathcal{A}\langle E_i, F_i, K_\mu\rangle : \text{De Concini-Kac form}$$

$$\mathrm{U}_{\mathcal{A}} = \mathcal{A}[\mathrm{Br}(W)]\langle \mathbf{E}_i = E_i/(q_i - q_i^{-1}), \mathbf{F}_i, K_\mu\rangle : \text{dual Lusztig form}$$

We focus on $\mathrm{U}_{\mathcal{A}}$. Set $\mathrm{U}_{\mathcal{A}}^{\imath} = \mathrm{U}^{\imath} \cap \mathrm{U}_{\mathcal{A}}$.

- $\mathrm{U}_{\mathcal{A}}$ is a quantum deformation of the Poisson algebra $\mathbb{C}[G^*]$.
- $\mathbb{C} \otimes_{q \mapsto \varepsilon} \mathrm{U}_{\mathcal{A}} \cong \mathbb{C} \otimes_{q \mapsto \varepsilon} \mathrm{U}_{\mathcal{A}}^{DK}$, where ϵ is the primitive ℓ-th root of unity, for $\ell > 3$ odd.

Semi-classical limits

In general, suppose R_q is an $\mathbb{C}[q, q^{-1}]$-algebra. Suppose R_q satisfies the condition

$$[f, g] = fg - gf \in (q-1)R_q, \qquad \forall f, g \in R_q. \tag{1}$$

Then $R = \mathbb{C} \otimes_{q \mapsto 1} R_q$ is a commutative Poisson algebra over $\mathbb{C}$, where

$$\{\overline{f}, \overline{g}\} = \overline{\frac{[f, g]}{q-1}}.$$

R is called the semi-classical limit of R_q.

Semi-classical limits of $\mathrm{U}_{\mathcal{A}}$

Let G be the simply connected semisimple group with Lie algebra $\mathfrak{g}$. Let (B^+, B^-) be a pair of Borel groups, such that $H = B^+ \cap B^-$ is the maximal torus. The dual Poisson-Lie group is

$$G^* = \{(b_1, b_2) \in B^+ \times B^- \mid \pi_H^+(b_1)\pi_H^-(b_2) = id\}.$$

It carries a canonical Poisson structure.

Theorem (De Concini–Procesi)

The algebra $U_{\mathcal{A}}$ satisfies the condition (1). Moreover one has the canonical isomorphism

$$\mathbb{C} \otimes_{q \mapsto 1} \mathrm{U}_{\mathcal{A}} \xrightarrow{\sim} \mathbb{C}[G^*]$$

as Poisson algebras.

Semi-classical limit of $\mathrm{U}^\imath_{\mathcal{A}}$

Hint: $\Delta : \mathrm{U}^\imath_{\mathcal{A}} \to \mathrm{U}^\imath_{\mathcal{A}} \otimes \mathrm{U}_{\mathcal{A}} \quad \rightsquigarrow \quad$ Poisson homogeneous space of G^*
Involution $\theta : \mathfrak{g} \to \mathfrak{g}$ integrates to $\theta : G \to G$. Recall
$G^* \subset B^+ \times B^- \subset G \times G$. Set

$$G_\theta = \{(g, \theta(g)) \mid g \in G\} \subset G \times G,$$

and $K^\perp \subset G^*$ to be the identity component of $G_\theta \cap G^*$.

Theorem (S., 2024)

The quotient $K^\perp \backslash G^*$ is an affine G^*-Poisson homogeneous space. Hence $\mathbb{C}[K^\perp \backslash G^*] \subset \mathbb{C}[G^*]$ is a Poisson subalgebra.
There is a canonical isomorphism

$$\mathbb{C} \otimes_{q \to 1} \mathrm{U}^\imath_{\mathcal{A}} \cong \mathbb{C}[K^\perp \backslash G^*]$$

as Poisson algebras.

Representation of $\mathrm{U}_{\mathcal{A}}$ at root of 1

Let $\ell > 3$ be an odd integer, and ε be a primitive ℓ-th root of 1. Let

$$\mathrm{U}_1 = \mathbb{C} \otimes_{q \mapsto 1} \mathrm{U}_{\mathcal{A}} \cong \mathbb{C}[G^*], \quad \mathrm{U}_\varepsilon = \mathbb{C} \otimes_{q \mapsto \varepsilon} \mathrm{U}_{\mathcal{A}}.$$

[De Concini–Kac–Procesi] constructed a Poisson algebra embedding

$$\mathsf{Fr} : \mathrm{U}_1 \to Z(\mathrm{U}_\varepsilon); \qquad \mathbf{E}_i \mapsto \mathbf{E}_i^\ell, \mathbf{F}_i \mapsto \mathbf{F}_i^\ell, \mathbf{K}_\mu \mapsto \mathbf{K}_{\ell\mu}.$$

$Z_0 = \mathsf{Fr}(\mathrm{U}_1)$ is called the Frobenius center. Moreover,

- U_ε is a free Z_0-module of rank $\ell^{\dim \mathfrak{g}}$.
- For $\chi \in G^*$, let $\mathrm{U}_{\varepsilon,\chi} = \mathrm{U}_\varepsilon / \mathfrak{m}_\chi \mathrm{U}_\varepsilon$. Then $\mathrm{Irr}\ \mathrm{U}_\varepsilon = \sqcup_{\chi \in G^*} \mathrm{Irr}\ \mathrm{U}_{\varepsilon,\chi}$.
- Let $\varphi : G^* \to G$ be $(b_1, b_2) \mapsto b_1^{-1} b_2$. Then $\mathrm{U}_{\varepsilon,\chi} \cong \mathrm{U}_{\varepsilon,\chi'}$, if $\varphi(\chi)$, $\varphi(\chi')$ are in the same conjugacy class of G ($\approx \chi$, χ' are in the same symplectic leaf).

$(\mathrm{U}_\varepsilon, Z_0)$ is a Poisson order in the sense of [Brown–Gordon].

Representation of $\mathrm{U}^\imath_{\mathcal{A}}$ at root of 1

Let

$$\mathrm{U}^\imath_1 = \mathbb{C} \otimes_{q\mapsto 1} \mathrm{U}_{\mathcal{A}} \cong \mathbb{C}[K^\perp \backslash G^*], \quad \mathrm{U}^\imath_\varepsilon = \mathbb{C} \otimes_{q\mapsto \varepsilon} \mathrm{U}_{\mathcal{A}}$$

Theorem (S.–Zhang, 2025+)

There is a Poisson algebra embedding

$$Fr^\imath : \mathrm{U}^\imath_1 \to Z(\mathrm{U}^\imath_\varepsilon).$$

We call $Z^\imath_0 = Fr^\imath(\mathrm{U}^\imath_1)$ the Frobenius center of $\mathrm{U}^\imath_\varepsilon$. Then $\mathrm{U}^\imath_\varepsilon$ is free over $Z^\imath_0$ of rank $\ell^{\dim \mathfrak{k}}$.

In split rank one, we have

$$\mathrm{Fr}^\imath(\mathbf{B}_i) = \mathbf{B}_i \prod_{r=1}^{(\ell-1)/2} (\mathbf{B}_i^2 + (q_i - q_i^{-1})^2 [2r-1]_i^2).$$

Representation of $\mathrm{U}^{\imath}_{\mathcal{A}}$ at root of 1, continued

We have central subalgebra $Z_0^{\imath} \subset \mathrm{U}^{\imath}_{\varepsilon}$, where Spec $Z_0^{\imath} \cong K^{\perp} \backslash G^*$. For $\chi \in K^{\perp}\backslash G^*$, set $\mathrm{U}^{\imath}_{\varepsilon,\chi} = \mathrm{U}^{\imath}_{\varepsilon}/\mathfrak{m}_{\chi}\mathrm{U}^{\imath}_{\varepsilon}$. Then

$$\mathrm{Irr}\ \mathrm{U}^{\imath}_{\varepsilon} = \sqcup_{\chi \in K^{\perp}\backslash G^*} \mathrm{Irr}\ \mathrm{U}^{\imath}_{\varepsilon,\chi}.$$

Let

$$\varphi^{\imath} : K^{\perp}\backslash G^* \to G, \quad K^{\perp}(b_1, b_2) \mapsto \theta(b_1)^{-1} b_2.$$

Theorem (S.–Zhang, 2025+)

For χ, χ' in $K^{\perp}\backslash G^*$, suppose $\varphi^{\imath}(\chi)$, $\varphi^{\imath}(\chi')$ are in the same θ-twisted conjugacy class, that is, $\varphi^{\imath}(\chi) = g\varphi^{\imath}(\chi')\theta(g)^{-1}$. Then $\mathrm{U}^{\imath}_{\varepsilon,\chi} \cong \mathrm{U}^{\imath}_{\varepsilon,\chi'}$ as $\mathbb{C}$-algebras.

In other words, we have a finite map

$$\psi : \mathrm{Irr}\ \mathrm{U}^{\imath}_{\varepsilon} \longrightarrow G$$

with isomorphic fibres along θ-twisted conjugacy classes.

More theorems

Theorem (S.–Zhang, 2025+)

- *The inclusion* $Z_0^\imath \hookrightarrow Z(\mathrm{U}_\varepsilon^\imath)$ *induces a finite map* $\mathrm{Spec}\ Z(\mathrm{U}_\varepsilon^\imath) \to K^\perp \backslash G^*$ *of degree* $\ell^{\mathrm{rank}\ \mathfrak{k}}$.
- *For* $V \in \mathrm{Irr}\ U_\varepsilon^\imath$, *we have* $\dim V \leq \ell^{|\Phi_{\mathfrak{k}}^+|}$.
- *Any generic irreducible representation of* $\mathrm{U}_\varepsilon^\imath$ *is a direct summand of an irreducible representation of* U_ε.

Conjecture

For any $\chi \in K^\perp \backslash G^*$ and $V \in \mathrm{Irr}\ \mathrm{U}_{\varepsilon,\chi}^\imath$. One has

$$\ell^{(\dim \mathcal{C}_\theta(\varphi^\imath(\chi)) - \dim \mathfrak{g} + \dim \mathfrak{k})/2} \mid \dim V.$$

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Thank you !