Kaluza-Klein AdS Virasoro-Shapiro and Veneziano Amplitudes

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AdS/CFT Meets Carrollian & Celestial Holography

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Bo Wang (王波), Di Wu (吴迪), EYY, arXiv:2503.01964

Bo Wang, arXiv:2508.14968

w.i.p. w/ Xiaoyu Fa (法肖羽), Bo Wang

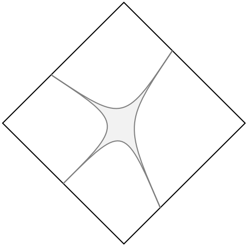
$$\underbrace{\text{Kaluza-Klein}}_{p} \quad \underbrace{\text{AdS}}_{R_{\text{AdS}}} \quad \underbrace{\begin{matrix}\text{Virasoro-Shapiro}\\ \text{Veneziano}\end{matrix}}_{l_{\text{s}}} \quad \underbrace{\text{(tree)}}_{1/c}$$

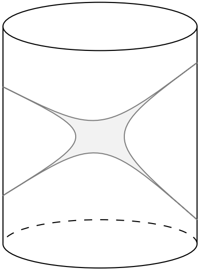
$$c = \frac{N^2-1}{2}$$

- Objects under study $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle \equiv \langle p_1 p_2 p_3 p_4\rangle$.

$$\mathcal{N}=4 \text{ SYM:} \quad \mathcal{O}_p \propto \text{tr}\big(\phi^{I_1}\phi^{I_2}\cdots\phi^{I_p}\big) y_{I_1} y_{I_2}\cdots y_{I_p}$$

Motivation



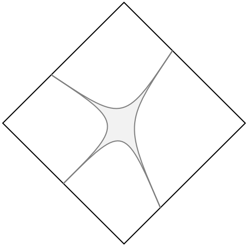


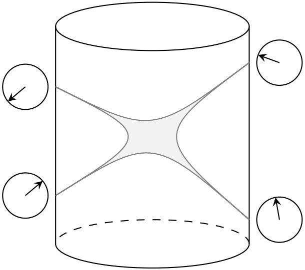
Virasoro-Shapiro
(finite l_s)

AdS Virasoro-Shapiro
$N \gg 1$ but finite λ (finite l_s and R_{AdS})

VS + curvature correction
$N \gg \lambda \gg 1$ (finite l_s but large R_{AS})

Motivation





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$N \gg 1$ but finite λ (finite l_s and R_{AdS})

VS + curvature correction
$N \gg \lambda \gg 1$ (finite l_s but large R_{AdS}, all KK)

$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$

Virasoro-Shapiro ($A^{(0)}$)

$$-\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

$(S+T+U=0)$

AdS Virasoro-Shapiro

$$\langle 2222 \rangle_{\text{int}} = \int [\mathrm{d}\gamma] M(s,t) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{(x_{ij}^2)^{\gamma_{ij}}}$$

low energy

$$M = M^{\text{SG}} + \lambda^{-\frac{3}{2}} M^3 + \cdots$$

Borel: $R_{\text{AdS}} \to \infty$

$$\frac{1}{STU} + 2\zeta_3 + 2\hat{\sigma}_2\zeta_5 + 2\hat{\sigma}_3\zeta_3^2 + \cdots$$

$(\hat{\sigma}_2 = S^2+T^2+U^2, \;\; \hat{\sigma}_3 = STU)$

$$A = \boxed{A^{(0)}} + \lambda^{-\frac{1}{2}} A^{(1)} + \cdots$$

$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$

- Low-energy expansion of Mellin
  (Fixed R_{AdS}, small ℓ_{s})

$$M(s,t) = \frac{8}{(s-2)(t-2)(u-2)} + \sum_{a,b=0}^{\infty} \frac{\#_{a,b}\,\sigma_2^a\sigma_3^b}{\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} \Big(\alpha_{a,b}^{(0)} + \frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{a,b}^{(2)}}{\lambda} + \cdots\Big)$$

$\sum p = 2\Sigma$, $s+t+u = 2\Sigma - 4$, $\sigma_2 = s^2+t^2+u^2$, $\sigma_3 = stu$.
Cannot be directly compared with Virasoro-Shapiro.

- Flat space limit via Borel transformed Mellin
  (Fixed ℓ_{s}, large R_{AdS})

$$\frac{A(S,T)}{\lambda^{\frac{3}{2}}\Gamma(\Sigma-1)} = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}\beta}{2\pi i} \frac{e^{\beta}}{\beta^{\Sigma+2}} M\Big(\frac{2\sqrt{\lambda}S}{\beta} + \frac{2\Sigma-4}{3}, \frac{2\sqrt{\lambda}T}{\beta} + \frac{2\Sigma-4}{3}\Big)$$

$$= \frac{1}{\lambda^{\frac{3}{2}}\Gamma(\Sigma-1)} \sum_{k=0}^{\infty} \lambda^{-\frac{k}{2}} A^{(k)}(S,T)$$

$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$

- Low-energy expansion of Mellin

$$M(s,t) = \frac{8}{(s-2)(t-2)(u-2)} + \sum_{a,b=0}^{\infty} \frac{\#_{a,b}\,\sigma_2^a\sigma_3^b}{\lambda^{\frac{3}{2}+a+\frac{3}{2}b}}\left(\alpha_{a,b}^{(0)} + \underbrace{\frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{a,b}^{(2)}}{\lambda} + \cdots}_{\text{curvature effects}}\right)$$

- Virasoro-Shapiro (determined by flat space result)

$$A^{(0)} = \frac{1}{STU} + 2\sum_{a,b=0}^{\infty} \alpha_{a,b}^{(0)}\,\hat{\sigma}_2^a\hat{\sigma}_3^b$$

- First curvature correction [Alday, Hansen, Silva, '22]
  (dispersive sum rules + single-valuedness)

$$A^{(1)} = -\frac{2\hat{\sigma}_2}{3\hat{\sigma}_3^2} + 2\sum_{a,b=0}^{\infty} \alpha_{a,b}^{(1)}\,\hat{\sigma}_2^a\hat{\sigma}_3^b$$

$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$

- The Mellin amplitude reads

$$
\begin{aligned}
M(s,t) =& \frac{8}{(s-2)(t-2)(u-2)} \\
&+ \lambda^{-\frac{3}{2}} 120\ \zeta_3 \\
&+ \lambda^{-\frac{5}{2}} (630\,\sigma_2 \qquad\qquad - 1890)\ \zeta_5 \\
&+ \lambda^{-\frac{6}{2}} (5040\,\sigma_3 - 1260\,\sigma_2 - 20160)\ \zeta_3^2 \\
&+ \lambda^{-\frac{7}{2}} (\#_1\sigma_2^2 \quad + \#_2\,\sigma_3 \quad + \#_3\sigma_2 + \cdots)\zeta_7 \\
&+ \mathcal{O}(\lambda^{-\frac{7}{2}})\,.
\end{aligned}
$$

- The corresponding AdS VS amplitudes are

$$
\begin{aligned}
A^{(0)}(S,T) &= \frac{1}{\hat{\sigma}_3} + 2\zeta_3 + \hat{\sigma}_2\zeta_5 + \hat{\sigma}_3\zeta_3^2 + \cdots \\
\mathcal{A}^{(1)}(S,T) &= -\frac{\hat{\sigma}_2}{3\hat{\sigma}_3^2} - \frac{22}{3}\hat{\sigma}_2\zeta_3^2 - \frac{537}{8}\hat{\sigma}_3\,\zeta_7 + \cdots
\end{aligned}
$$

$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$

- The Wilson coefficients of $A^{(0)}$ and $A^{(1)}$ lie in the ring of single-valued multiple zeta values (SVMZV) [Brown, '13]

$$\zeta^{\mathrm{sv}}_{s_1,\cdots,s_k} = \sum n_1^{-s_1} \cdots n_k^{-s_k}, \quad n_1 > \cdots > n_k > 0.$$

In particular, $\zeta^{\mathrm{sv}}_{2n} = 0$, $\zeta^{\mathrm{sv}}_{2n+1} = 2\,\zeta_{2n+1}$.

- SVMZV naturally arise as boundary values of single-valued multiple polylogarithms (SVMPL)

$$\zeta^{\mathrm{sv}}_{s_1,\cdots,s_k} = \mathcal{L}_{0^{s_1-1}10^{s_2-1}\ldots 0^{s_k-1}1}(1).$$

- SVMPL are recursively defined by differential relations

$$\partial_z \mathcal{L}_{0w}(z) = z^{-1}\mathcal{L}_w(z) \quad \partial_z \mathcal{L}_{1w}(z) = (z-1)^{-1}\mathcal{L}_w(z),$$

with $\mathcal{L}_\emptyset(z) = 1$ and $\mathcal{L}_{0^n}(z) = (n!)^{-1}\log^n|z|^2$ [Brown, '20].

- These naturally arise from string worldsheets!

$\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$

- Worldsheet integral for Virasoro-Shapiro $A^{(0)}$

$$A^{(0)} = \int \mathrm{d}^2 z \, |z|^{-2S-2} |1-z|^{-2T-2} \, \frac{1}{U^2} \, .$$

- Worldsheet integral for $A^{(1)}$ [Alday, Hansen, Silva, '23]

$$A^{(1)}(S,T) = B^{(1)}(S,T) + B^{(1)}(U,T) + B^{(1)}(S,U),$$
$$B^{(1)}(S,T) = \int \mathrm{d}^2 z \, |z|^{-2S-2} |1-z|^{-2T-2} \, G^{(1)}(z,\bar{z}).$$

- It turns out

$$G^{(1)} = F_0(\mathcal{L}_{000} + \mathcal{L}_{111}) + F_1(\mathcal{L}_{010} + \mathcal{L}_{101}) + F_2(\mathcal{L}_{001} + \mathcal{L}_{110}) + (\text{redundancy}).$$

$$F_0 = \frac{2S}{3U}, \quad F_1 = \frac{S+5T}{6U}, \quad F_2 = \frac{2(T-S)}{3U}.$$

Bootstrap based on the worldsheet

- # of SVMPLs with fixed weight and singularities is finite.
  $\Rightarrow$ ansatz for the worldsheet integrand.
- Applied to $A^{(2)}$ of $\langle 2222 \rangle$ [Alday, Hansen, '23] and $A^{(1)}$ of $\langle 22pp \rangle$ [Fardlli, Hansen, Silva, '23].
- Case of $\langle 22pp \rangle$

$$A^{(1)}(S,T) = B_1^{(1)}(S,T) + B_1^{(1)}(S,U) + B_1^{(1)}(U,T) + B_2^{(1)}(S,T) + B_2^{(1)}(S,U).$$

$$G_i^{(1)} = \sum_{u=1}^{4} F_{i,u}^{(1)+} \mathcal{L}_u^+ + \sum_{u=1}^{3} F_{i,u}^{(1)-} \mathcal{L}_u^- + (\text{redundancy}),$$

$$F_1^{(1)+} = \frac{1}{24}(-p^2, 2(p-2)p, p^2-2p-6, 48),$$

$$F_1^{(1)-} = \frac{p^2(T-S)}{24U}(-1,2,1),$$

$$F_2^{(1)+} = \frac{-p(p-2)}{24U}(3S, -2(2S+T), -2S-T, 0),$$

$$F_2^{(1)-} = \frac{-p(p-2)}{24U}(3S, -2(2S-T), -2S+T).$$

- ✓ The use of worldsheet integrals naturally packages stringy corrections at different levels.
- ✗ In principle, we can apply this to arbitrary configuration of Kaluza-Klein charges.
  However, this is a *daunting task*.

Past experience tells that
we'd better package all objects in the same class.

[Caron-Huot, Trinh, '18] [Caron-Huot, Coronado, '21]

$$\mathbb{O} = \sum_{p=2}^{\infty} \#_p \, \mathcal{O}_p .$$

The AdS×S formalism

- Ordinary Mellin [Mack, '09]

$$\langle p_1 p_2 p_3 p_4 \rangle_{\rm int} = \int [{\rm d}\gamma]\, M(s,t;y_{ij}) \prod_{i<j} \frac{\Gamma(\gamma_{ij})}{(x_{ij}^2)^{\gamma_{ij}}}.$$

$\gamma_{ij} = \gamma_{ji}$, $\sum_{j\neq i} \gamma_{ij} = p_i$, $s = p_1 + p_2 - 2\gamma_{12}$, etc.

- Mellin on the sphere [Aprile, Vieira, '20]

$$M(s,t;y_{ij}) = \sum_{n_{ij}} \mathcal{M}(s,t;n_{ij}) \prod_{i<j} \frac{y_{ij}^{n_{ij}}}{\Gamma(n_{ij}+1)} \prod_i \delta_{p_i - 2, \sum_{j\neq i} n_{ij}}.$$

$n_{ij} = n_{ji}$, $\sum_{j\neq i} n_{ij} = p_i - 2$.

- For $\langle \mathbb{OOOO} \rangle$ we simply do NOT constrain n_{ij}.

The AdS×S formalism

Numerous applications. To name a few:

- ☐ Mellin for sugra [Aprile, Vieira, '20]

$$\mathcal{M}^{\text{SG}} = -\frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}.$$

  $\mathbf{s} = \frac{1}{2}(s - p_1 - p_2) + n_{12}$, $\mathbf{t} = \frac{1}{2}(t - p_1 - p_4) + n_{14}$, $\mathbf{s} + \mathbf{t} + \mathbf{u} = -4$.
  Manifest 10d structure.

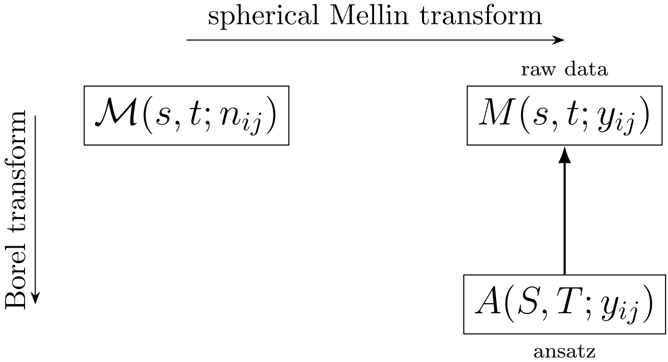
- ☐ AdS stringy correction at $\lambda^{-3/2}$ [Aprile, Vieira, '20]

$$\mathcal{M}^3 = 2(\Sigma - 1)_3\,\zeta_3.$$

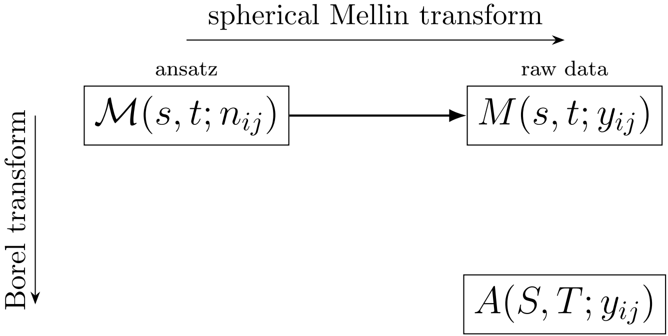
  Also [Abl, Heslop, Lipstein, '20] [Aprile, Drummond, Paul, Santagata, '20].

- ☐ Tree-level five-point gluon scattering in $\text{AdS}_5 \times \text{S}^3$
  [Huang, Wang, **EYY**, Zhang, '24].
- ☐ Tree-level $\langle pqrst \rangle$
  [Fernandes, Goncalves, Huang, Tang, Vilas Boas, **EYY**, 2507.14124].

Strategy


spherical Mellin transform
raw data
$\mathcal{M}(s,t;n_{ij})$
$M(s,t;y_{ij})$
Borel transform
$A(S,T;y_{ij})$
ansatz

Strategy


spherical Mellin transform
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raw data
$\mathcal{M}(s,t;n_{ij})$
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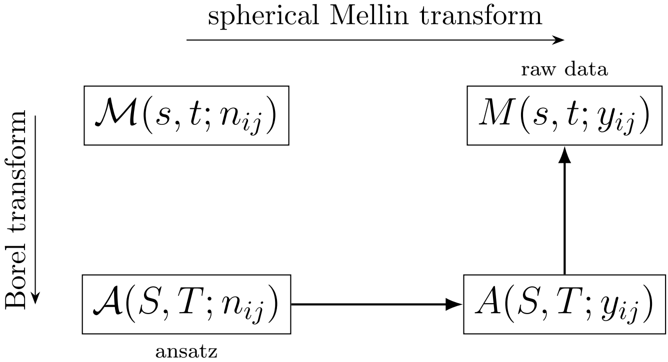
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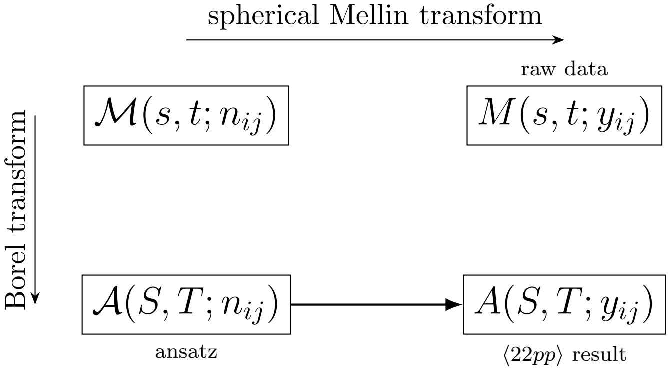
ansatz

Strategy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raw data
$\mathcal{M}(s,t;n_{ij})$
$M(s,t;y_{ij})$
Borel transform
$\mathcal{A}(S,T;n_{ij})$
$A(S,T;y_{ij})$
ansatz
$\langle 22pp \rangle$ result

Worldsheet ansatz

- A quick glimpse at curvature correction to sugra

$$\mathcal{M}^{\rm SG} \to \mathcal{A}^{(1),\rm SG} = \frac{2-\Sigma}{6\,\hat{\sigma}_3^2}\Big(3(n_s S^2 + n_t T^2 + n_u U^2) - (n_s + n_t + n_u - 1)(S^2 + T^2 + U^2)\Big),$$

where

$$n_s = n_{12} + n_{34}, \quad n_t = n_{14} + n_{23}, \quad n_u = n_{13} + n_{24}.$$

Hints on ingredients to appear in the ansatz.

- Ansatz in terms of worldsheet integrals

$$\mathcal{A}^{(1)}(S,T) = \mathcal{B}^{(1)}(S,T;n_s,n_t) + \mathcal{B}^{(1)}(U,T;n_u,n_t) + \mathcal{B}^{(1)}(S,U;n_s,n_u)\,,$$

$$\mathcal{B}^{(1)}(S,T;n_s,n_t) = \int d^2z\,|z|^{-2S-2}|1-z|^{-2T-2}\,\mathcal{G}^{(1)}(z,\bar{z};n_s,n_t)\,.$$

Worldsheet ansatz

- Worldsheet integrand contains SVMPLs at weight 3, with singularities only at $z = 0, 1, \infty$

$$\mathcal{L}_i^+ = \left(\mathcal{L}_{000}^+, \mathcal{L}_{001}^+, \mathcal{L}_{010}^+, \zeta_3\right) ,$$
$$\mathcal{L}_i^- = \left(\mathcal{L}_{000}^-, \mathcal{L}_{001}^-, \mathcal{L}_{010}^-\right) .$$

  $\mathcal{L}_i^\pm$ is symmetric/anti-symmetric under $z \leftrightarrow 1 - z$, e.g.,

$$\mathcal{L}_{000}^+ = \mathcal{L}_{000} + \mathcal{L}_{111} = \frac{\log^3 |z|^2}{3!} + \frac{\log^3 |1-z|^2}{3!} .$$

- Build the integrand by linear combination

$$\mathcal{G}^{(1)}(z, \bar{z}; n_s, n_t) = \sum_{i=1}^{4} \mathcal{F}_i^+ \mathcal{L}_i^+ + \sum_{j=1}^{3} \mathcal{F}_j^- \mathcal{L}_j^- .$$

  The use of SVMPLs guarantees that the amplitude only contains single-valued MZVs.

Worldsheet ansatz

- The prefactors $\mathcal{F}_i^{\pm}(S,T)$ are homogeneous functions with a universal denominator U^2 and enjoy the general structure

$$\mathcal{F}_i^{\pm}(S,T) = \frac{f_1^{\pm} S^2 + f_2^{\pm} \, S\, T + f_3^{\pm} \, T^2}{U^2} \,,$$

where $f_i^{\pm}$ are linear combinations of

$$\{\Sigma \, n_s, \Sigma \, n_t, \Sigma \, n_u, \Sigma \,, n_s, n_t, n_u\}$$

with unknown coefficients.

- Degree and denominator motivated by $A^{(0)}$ and $A^{(1)}_{22pp}$.
- Parameters motivated by the structure of $\mathcal{A}^{(1),\mathrm{SG}}$.

$$\mathcal{A}^{(1),\mathrm{SG}} = \frac{2-\Sigma}{6\,\hat{\sigma}_3^2}\big(3(n_s S^2 + n_t T^2 + n_u U^2) \\ - (n_s + n_t + n_u - 1)(S^2 + T^2 + U^2)\big),$$

Consistency constraints for the bootstrap

- Crossing symmetry. Invariance under $z \leftrightarrow 1-z$, $n_s \leftrightarrow n_t$

$$\mathcal{B}^{(1)}(S,T;\, n_s,n_t) = \mathcal{B}^{(1)}(T,S;\, n_t,n_s)\,.$$

- Recover supergravity contributions $\mathcal{A}^{(1),\mathrm{SG}}$.
- After Mellin transformation on the sphere, match the known $A^{(1)}$ for $\langle 22pp\rangle$.

Results

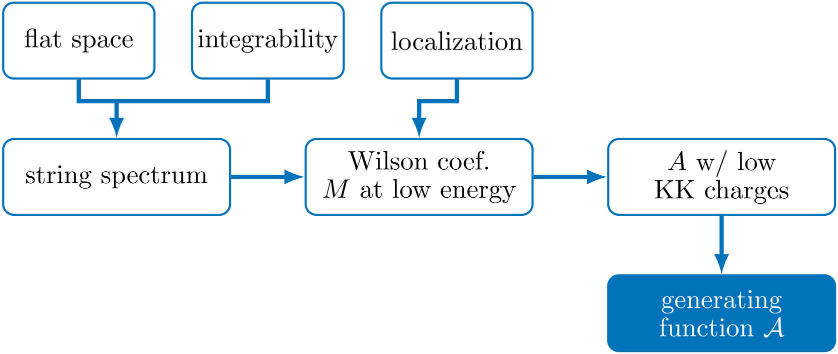
- The final result (apart from redundancies) only depends on five linearly independent SVMPLs

$$\mathcal{L}_{1/4} = \mathcal{L}^{+/-}_{000} \,, \quad \mathcal{L}_2 = -\mathcal{L}^{+}_{010} + 4\zeta_3 \,,$$
$$\mathcal{L}_3 = \mathcal{L}^{+}_{000} - \mathcal{L}^{+}_{001} - \mathcal{L}^{+}_{010} \,,$$
$$\mathcal{L}_5 = \mathcal{L}^{-}_{000} - \mathcal{L}^{-}_{001} - \mathcal{L}^{-}_{010} \,.$$

- In terms of this basis

$$\mathcal{G}^{(1)}(z,\bar{z};n_s,n_t) = (\Sigma - 2) \times \sum_{i=1}^{5} \mathcal{F}_i \mathcal{L}_i \,,$$
$$\mathcal{F}_1 = \frac{(n_s - n_t)(S-T) + 2U}{12(S+T)} \,, \quad \mathcal{F}_2 = \frac{1}{2}\frac{1}{\Sigma - 2} \,,$$
$$\mathcal{F}_3 = \frac{n_s\, S + n_t\, T + n_u\, U}{12(S+T)} \,, \; \mathcal{F}_4 = \frac{n_s - n_t}{12} \,,$$
$$\mathcal{F}_5 = \frac{n_s S - n_t T + (n_u + 2)(T - S)}{12(S+T)} \,.$$

Skip the analysis at low energy



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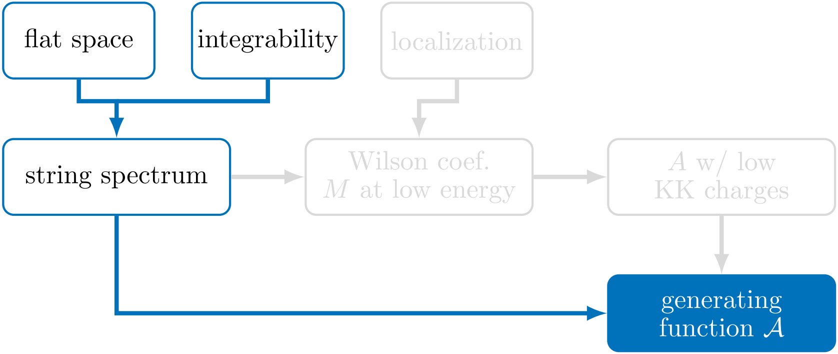


$\mathcal{A}^{(1)}$ in AdS Veneziano amplitude

[Bo Wang, arXiv:2508.14968]

(on job market this year)

OPE in Mellin space

- (Super-)conformal block expansion

$$\underbrace{\mathcal{I}^{-1}\langle p_1 p_2 p_3 p_4 \rangle_{\text{int}}}_{\text{reduced correlator}} = \sum_{\mathcal{O}} f_{\mathcal{O}} G_{\tau+2,\ell}(z, \bar{z})$$

  - $\mathcal{O}$: single-trace conformal primary $\leftrightarrow$ string excitations.
  - $f_{\mathcal{O}}$ also captures R-symmetries.

- In Mellin space

$$M(s,t) \sim \sum_{\mathcal{O}} \sum_{m=0}^{\infty} \frac{C_{\mathcal{O}}}{s - \tau - 2m} \underbrace{Q_{\ell,m}^{\tau+2}(u - p_1 - p_4)}_{\text{Mack polynomial}}.$$

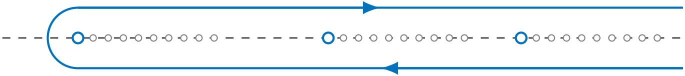
  - $C_{\mathcal{O}} \propto f_{\mathcal{O}}$.
  - m labels conformal descendants.

String spectrum near flat space limit

- At large λ

$$\tau = \sqrt{\delta}\,\lambda^{\frac{1}{4}} + \tau_1 + \tau_2\,\lambda^{-\frac{1}{4}} + \cdots\,, \quad \delta = 1, 2, \ldots$$
$$f_{\mathcal{O}} = f_0 + f_1\,\lambda^{-\frac{1}{4}} + f_2\,\lambda^{-\frac{1}{2}} + \cdots\,.$$

- Borel transform



Descendant poles condensate: $\sum_{m=0}^{\infty} \mapsto \int_0^{\infty} \mathrm{d}x\,\tau^2$

- In consequence

$$\frac{A(S,T)}{\lambda\Gamma(\Sigma-2)} \sim \sum_{\mathcal{O}} \int_0^{\infty} \frac{\mathrm{d}x\,\tau^2}{4\sqrt{\lambda}S}\,\frac{e^{\beta_*}}{\beta_*^{\Sigma-2}}\,C_{\mathcal{O}} Q^{\tau+2}_{\ell,x\tau^2}\left(\frac{\sqrt{\lambda}U}{2\beta_*} + \frac{2\Sigma-2}{3} - p_1 - p_4\right)$$

- At each order in the λ expansion, Q decompose into Gegenbauer C_l and its derivatives.

Stringy spectrum near flat space limit

- x integral creates poles in $(S-\delta)$ (now switch to $\mathcal{A}$)

$$\mathcal{A}^{(k)}(S,T) \sim \sum_{i=1}^{3k+1} \frac{\mathcal{R}_i^{(k)}(\delta,T)}{(S-\delta)^i} + (\text{regular})$$

- In $\mathcal{A}^{(0)}$ (from Veneziano amplitude)

$$\mathcal{R}_1^{(0)} = -\sum_{l=0}^{\delta-1} \langle f_0\rangle_{\delta,\ell}\, \frac{C_\ell(\frac{2T}{\delta}+1)}{\delta(\ell+1)}$$

- In $\mathcal{A}^{(1)}$ (# are polynomials in δ and T, and $C^{(k)} = \partial_T^k C_\ell$)

$$\mathcal{R}_4^{(1)} = \sum \langle f_0\rangle_{\delta,\ell}\, \delta\, \# C_\ell^{(0)}, \qquad \mathcal{R}_3^{(1)} = \sum \langle f_0\rangle_{\delta,\ell} (\# C^{(1)} + \# C^{(0)}),$$

$$\mathcal{R}_2^{(1)} = \sum \langle f_0\rangle_{\delta,\ell}\, \delta^{-1} (\# C^{(2)} + \# C^{(1)} + \# C^{(0)}) + \sum \langle f_0 \tau_2\rangle_{\delta,\ell}\, \delta^{-\frac{1}{2}}\, \# C^{(0)},$$

$$\mathcal{R}_1^{(1)} = \sum \langle f_0\rangle_{\delta,\ell}\, \delta^{-2} (\# C^{(3)} + \# C^{(2)} + \# C^{(1)} + \# C^{(0)}) + \sum \langle f_0 \tau_2\rangle_{\delta,\ell}\, \delta^{-\frac{3}{2}} (\# C^{(1)} + \# C^{(0)}) + \sum \langle f_2\rangle_{\delta,\ell}\, \delta^{-1}\, \# C^{(0)}.$$

Stringy pole from the worldsheet

- Worldsheet ansatz (note Veneziano is planar)

$$\mathcal{A}^{(k)} = -\frac{1}{U}\int_0^1 \mathrm{d}z\, z^{-S-1}(1-z)^{-T-1}\,\mathcal{G}^{(k)}(S,T;z;n_{ij}),$$
$$\mathcal{G}^{(1)} = \sum_i \sum_\pm \mathcal{F}_i^\pm(S,T;n_{ij})\mathcal{L}_i^\pm(z)$$

  Now $\mathcal{L}_i^\pm$ include both weight 3 and lower weight SVMPLs.

- Explore s-channel OPE, $z \to 0$: $\mathcal{L}_w \sim \sum \#\, z^{\#}\, \log^{\#}(z)$

$$\int_0^1 \mathrm{d}z\, z^{-S+\delta-1}\log^{k-1}(z) = -\frac{\Gamma(k+1)}{(S-\delta)^k}$$

  This helps derive $\mathcal{R}_k^{(1)}$ from the ansatz.

- Matching the known $\langle f_0\rangle_{\delta,\ell}$ completely fixes the ansatz, which further determines $\langle f_0\tau_2\rangle_{\delta,\ell}$ and $\langle f_2\rangle_{\delta,\ell}$!
- This strategy works similarly for Virasoro-Shapiro $\mathcal{A}^{(1)}$.

Comments on $\mathcal{A}^{(2)}$

- Only $A^{(2)}_{2222}$ has been worked out [Alday, Hansen, '23].
- Function basis at weight 6 for Virasoro-Shapiro $\mathcal{A}^{(2)}$.
- New data include $\langle f_0 \tau_2^2 \rangle_{\delta,\ell}$, $\langle f_0 \tau_4 + f_2 \tau_2 \rangle_{\delta,\ell}$, $\langle f_4 \rangle_{\delta,\ell}$, appearing only in $(S-\delta)^{-(k\leq 3)}$.
- Constrain the ansatz using data of stringy operators from both $\mathcal{A}^{(0)}$ and $\mathcal{A}^{(1)}$.

Discussion

- We establishes a framework for computing curvature corrections to the AdS Virasoro-Shapiro amplitude with arbitrary KK modes using AdS×S formalism.
- Worldsheet ansatz easily constrained by CFT data for massive stringy operators at lower order, yielding predictions at higher order.
- Interesting to generalize our method to higher-order corrections ($\mathcal{A}^{(2)}$ in progress) as well as other AdS background (e.g., [Jiang, Zhong, '25]).
- Origin of the worldsheet in curvature corrections (e.g., [Alday, Giribet, Hansen, '24]).

Thanks for your attention!

Questions & comments are welcome.