Hermite Orthogonality from Non-commutative KP Wave Functions

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TLDR: I started out in integrable systems, with emphasis on bispectrality.

Recently started on (exceptional) orthogonal polynomials, which seemed different.
However, those two topics are turning out to be more closely related than I thought.

The KP Hierarchy is a natural context for studying (exceptional) Hermite OPs, and perhaps all classical/exceptional OPs.

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- G. Wilson coined the term "adelic Grassmannian" for
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- (Okay...wake up now, I'm back to the important stuff.)

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- For $(X, Z) \in CM_1^n$ the eigenvalues q_j of $X - 2tZ$ move according to the Hamiltonian:

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- The involution which linearizes the motion of this integrable system (see Calogero, Moser, Kazhdan-Kostant-Sternberg and Gibbons-Hermensen, Krichever-Babelon-Billey-Talon) is $(X, Z) \mapsto (Z^\top, X^\top)$.

(nc)KP Wave Functions

- To $(X, Z) \in CM_r^n$ associate the function
$$\psi_{(X,Z)}(\vec{t}, z) = \left(I_r + a^\top(-X + \sum jt_j Z^{j-1})^{-1}(zI - Z)^{-1} b\right) e^{\sum t_j z^j}.$$

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- Then if $\psi = \psi_{(X,Z)} = We^{\sum t_j z^j}$ and $\mathscr{L} = W \circ \partial_{t_1} \circ W^{-1}$ one has
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The coefficients are $r \times r$ matrices. So, it is ncKP if $r > 1$.

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The map that
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swapping variables and transposing:
"bispectral involution".

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Wilson Invent.Math. 1998, Bergvelt-K-Gekhtman MPAG 2009

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- As a consequence we get *bispectrality:* $L_x \psi = p(z)\psi \qquad \psi\Lambda_z = \pi(x)\psi$

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If each distribution has support at one point, you get the same wave functions as CM_1^n.

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But you can also get *more*, like $\boldsymbol{n}$-soliton wave functions.

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- I spoke on this at NEEDS 2015 in Sardinia. Rob Milson approached me seeking help with bispectrality of ***exceptional orthogonal polynomials.***

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- But, those are not only Hermite orthogonal functions I care about!

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- The corresponding family of rational "Exceptional Hermites":

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Exceptional Hermites

Gómez-Ullate, Grandati, Milson 2013

- (Caveat: This is *my* alternative but equivalent description.)
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition s.t. $\lambda_j \geq \lambda_{j+1} \geq 0$ and $\sum \lambda_j = n$.
- The corresponding family of rational "Exceptional Hermites":

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There was no KP, no CM matrices. It was just straightforward bispectral theory with no obvious integrability connections.

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XHermites hiding in KP Wave Functions!

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There were no CM matrices. Recall the alternative method for wave functions using finitely-supported distributions? Remember the derivatives in the formulas for the XHermites? That was the connection.

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But, I thought the CM matrix approach was likely to be useful for this latest project. Thanks to Luke Paluso who "translated" the result into the CM matrix language for his master's thesis.

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This was just the KP time variable $t_2 = y$ in the paper. It acts as a scaling of the Hermite weight and at the time it seemed any value was as good as any other.

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Step 1 of 3: Orthogonality via Generating Functions

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- Lemma: The coefficients in the generating function $g(x,z)$ are orthogonal iff $\langle g(x,w), g(x,z) \rangle = \alpha(wz)$ is a function of the product "wz".

- So $\displaystyle\int_{-\infty}^{\infty} \hat{\psi}_{(X_\lambda, Z_\lambda)}(x, -1/2, w) \hat{\psi}_{(X_\lambda, Z_\lambda)}(x, -1/2, z) e^{-x^2/2} \, dx = \alpha(wz)$. Why?

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- It was my intention to address (a) using the bispectral involution and worry about (b) afterwards.

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just becomes

$$\oint \psi_{(X,Z)}(x,0,0,\ldots,w)\psi_{(X^\star,Z^\star)}(x,0,0,z)\,dx = 0.$$

Where $(X^\star, Z^\star)$ has some messy formula. Not so useful.

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- But, *miraculously* if $t_1 = x,\ t_2 = -1/2,\ t_j = 0,\ j \geq 3$

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True for every $(X,Z) \in CM_1^n$. (On the right track!)

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- Let $(X, Z) \in CM_r^n$. Then we have a nice formula for
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Recall

$$\psi_{(X,Z)}(\vec{t}, z) = \left(I_r + a^{\top}(-X + \sum jt_jZ^{j-1})^{-1}(zI - Z)^{-1}b\right) e^{\sum t_j z^j}.$$

Write out the integrand as a matrix product.

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Write out the integrand as a matrix product.

If X, Z, a, b are just randomly chosen matrices, the integrand will almost certainly not have an elementary anti-derivative with respect to x.

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But, use $[X,Z]-I = ba^{\top}$ to move all of X's to the right of all Z's, and get something for which you can just read off an AD. Involves matrix mult, exp, and erf.

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There are poles for $x \in \mathbb{R}$. But, we don't need to worry about that because all of the residues are zero. (Cf. Haese-Hill/Hallnäs/Veselov, *Lett.Math.Phys.* 2016).

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- We already knew something special happened for (X_λ, Z_λ) wrt the Hermite product, but this works for *every* CM pair and even in higher rank. (See next slide for what it tells us.)
- BTW This follows from a surprising fact: One can choose the roots of $\tau_{(X,Z)}(x,0,0,\ldots)$ and $\tau_{(Z^{\top},X^{\top})}(x,0,0,\ldots)$ independently. But $\tau_{(X,Z)}(x,-1/2,0,\ldots) = \tau_{(Z^{\top},X^{\top})}(x,-1/2,0,\ldots)$ is always true!

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So, we get a lot of Hermite orthogonality from KP Wave Functions and Bispectrality!

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- **Exceptional Hermites Rediscovered:** If $r = 1$, the latter holds exactly when $(X,Z) = (X_\lambda, Z_\lambda)$ for some partition.

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- There is also a link to Exceptional Krall polynomials (see forthcoming K-Milson in *J.Approx.Th.*) But no time for that now.