

Hermite Orthogonality from Non-commutative KP Wave Functions

ICMS June 2026

TLDR: I started out in integrable systems, with emphasis on bispectrality.

Recently started on (exceptional) orthogonal polynomials, which seemed different.

However, those two topics are turning out to be more closely related than I thought.

The KP Hierarchy is a natural context for studying (exceptional) Hermite OPs, and perhaps all classical/exceptional OPs.

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- We are generally only interested in these pairs up to simultaneous conjugation: $(X, Z) \sim (GXG^{-1}, GZG^{-1})$.

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- (Okay...wake up now, I'm back to the important stuff.)

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- For $(X, Z) \in CM_1^n$ the eigenvalues q_j of $X - 2tZ$ move according to the Hamiltonian:

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- The involution which linearizes the motion of this integrable system (see Calogero, Moser, Kazhdan-Kostant-Sternberg and Gibbons-Hermensen, Krichever-Babelon-Billey-Talon) is $(X, Z) \mapsto (Z^\top, X^\top)$.

(nc)KP Wave Functions

- To $(X, Z) \in CM_r^n$ associate the function

$$\psi_{(X,Z)}(\vec{t}, z) = \left(I_r + a^\top (-X + \sum j t_j Z^{j-1})^{-1} (zI - Z)^{-1} b \right) e^{\sum t_j z^j}.$$

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The coefficients are
 $r \times r$ matrices. So, it is ncKP
if $r > 1$.

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The map that linearizes CM has the effect of swapping variables and transposing: “bispectral involution”.

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Wilson Invent.Math. 1998,
Bergvelt-K-Gekhtman
MPAG 2009

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- As a consequence we get *bispectrality*:

$$L_x \psi = p(z) \psi \quad \psi \Lambda_z = \pi(x) \psi$$

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- But, I did not use CM matrices! Instead, used the other common construction for the same wave functions, finitely-supported distributions:

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If each distribution has support at one point, you get the same wave functions as $\mathbf{CM}_1^n$.

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But you can also get *more*, like ***n***-soliton wave functions.

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- I spoke on this at NEEDS 2015 in Sardinia. Rob Milson approached me seeking help with bispectrality of *exceptional orthogonal polynomials*.

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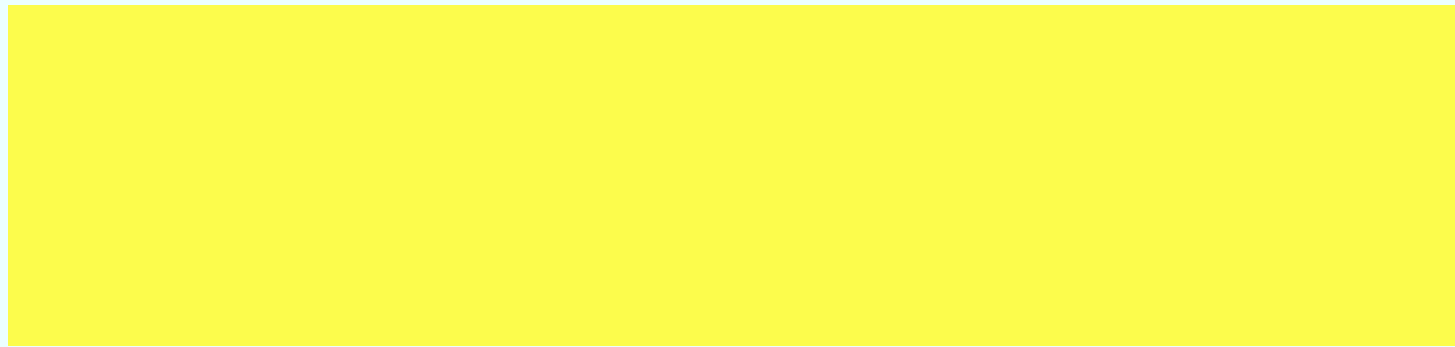
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- But, those are not only Hermite orthogonal functions I care about!

Exceptional Hermites

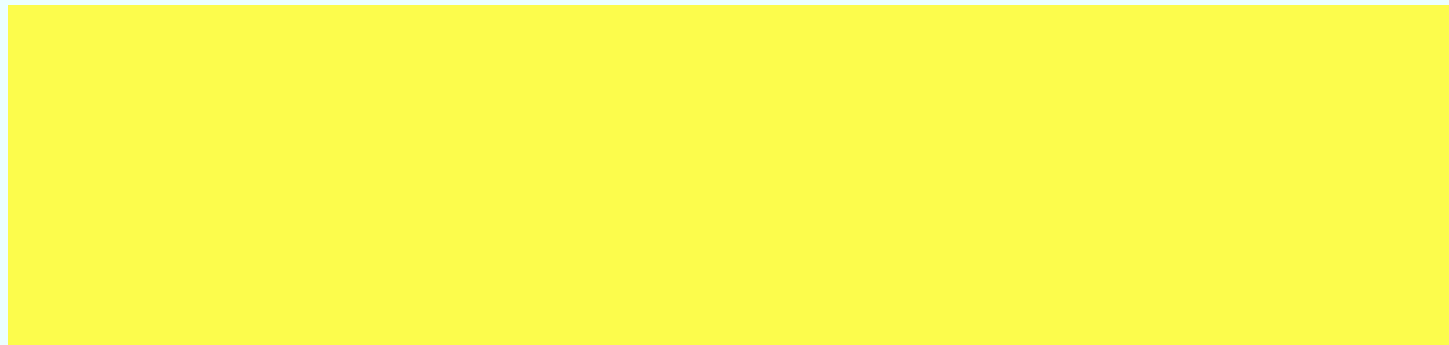
Gómez-Ullate, Grandati, Milson 2013



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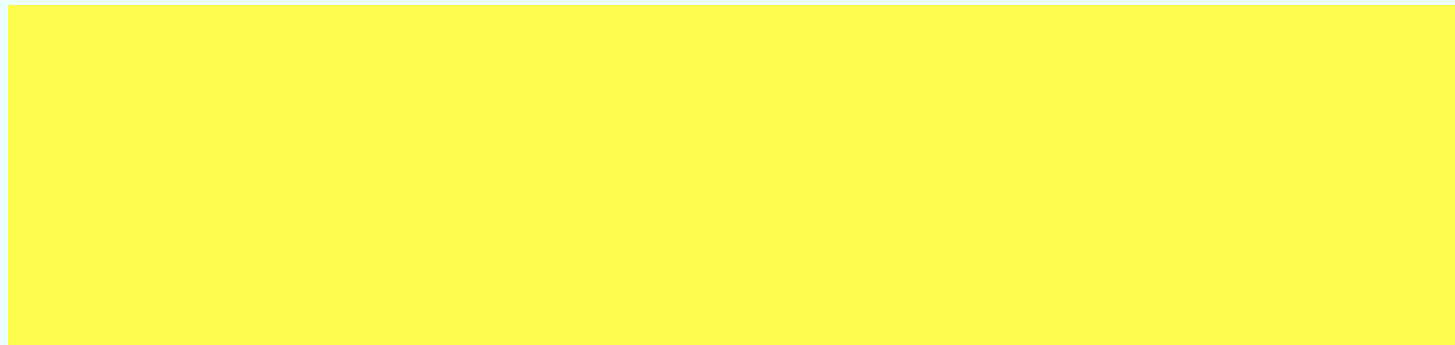
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- Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition s.t. $\lambda_j \geq \lambda_{j+1} \geq 0$ and $\sum \lambda_j = n$.
- The corresponding family of rational “Exceptional Hermites”:

$$\hat{h}_i^\lambda(x) = \frac{\text{Wr} \left(h_{\lambda_1-1+n}, h_{\lambda_2-2+n}, \dots, h_{\lambda_n-n+n}, h_i \right)}{\text{Wr} \left(h_{\lambda_1-1+n}, h_{\lambda_2-2+n}, \dots, h_{\lambda_n-n+n} \right)} = \frac{\hat{p}_i^\lambda(x)}{\tau_\lambda(x)}.$$

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- Note that $\hat{h}_i^\lambda(x) = 0$ if $i = \lambda_j - j + n$, but they are still a Hilbert basis for $\mathbb{C}[x]/\tau_\lambda(x)$ and still Hermite orthogonal $\left\langle \hat{h}_i^\lambda, \hat{h}_j^\lambda \right\rangle_H = c_i \delta_{ij}$.

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- Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition s.t. $\lambda_j \geq \lambda_{j+1} \geq 0$ and $\sum \lambda_j = n$.
- The corresponding family of rational “Exceptional Hermites”:

$$\hat{h}_i^\lambda(x) = \frac{\text{Wr} \left(h_{\lambda_1-1+n}, h_{\lambda_2-2+n}, \dots, h_{\lambda_n-n+n}, h_i \right)}{\text{Wr} \left(h_{\lambda_1-1+n}, h_{\lambda_2-2+n}, \dots, h_{\lambda_n-n+n} \right)} = \frac{\hat{p}_i^\lambda(x)}{\tau_\lambda(x)}.$$

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- They are also eigenfunctions for a 2nd order differential operator.
- But, what about the *recursion relation*? That’s what Job Milson et al wanted me to help with. And, we did it using “bispectral Darboux transformations”.

There was no KP, no CM matrices. It was just straightforward bispectral theory with no obvious integrability connections.

(Gómez-Ullate, -K-Kuijlaars-Milson *J.Approx.Th.* 2016)

XHermite's hiding in KP Wave Functions!



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In other words, generating functions for XHermite are KP wave functions.

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- Moreover, the KP flows and CM particles both played roles. And, the recursion relations could be easily produced $\Lambda_z \mapsto \hat{\Omega}_i$.

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- Note 1: $\hat{\psi}_{(X, Z)}(\vec{t}, z) = \det(zI - Z) \psi_{(X, Z)}(\vec{t}, z)$.
- Note 2: The actual statement in the paper was very different. Let me just quickly run through some of the things I changed for the talk.

XHermite hidden connections!

There were no CM matrices. Recall the alternative method for wave functions using finitely-supported distributions? Remember the derivatives in the formulas for the XHermite? That was the connection.

- So, Rob and I found the connection for the XHermite.
- ...but no analogy was needed. It was just a connection.

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XHermite's hidden connections!

But, I thought the CM matrix approach was likely to be useful for this latest project. Thanks to Luke Paluso who “translated” the result into the CM matrix language for his master’s thesis.

- So, Rob and I found a connection for the CM matrix approach.
- ...but no analogy was needed. It was just a direct translation.

- **Thm (K-Milson MPAG 2020):** For any λ there is $(X_\lambda, Z_\lambda) \in CM_1^n$ so that

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In other words, generating functions for XHermite's are KP wave functions.

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XHermite functions hiding in KP Wave Functions!

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In other words, the XHermite functions are KP wave functions.

- This was just the KP time variable $t_2 = y$ in the paper. It acts as a scaling of the Hermite weight and at the time it seemed any value was as good as any other. And, the

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In other words, the XHermite functions are KP wave functions.

- I would specialize to $t_2 = y = -1/2$ later for the new result. So, to keep things simple, I'm considering that case here.

- Note 1: $\hat{\psi}_{(X, Z)}(t, z) = \hat{\psi}_{(X, Z)}(t, z)$.

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In other words, some XHermite functions are KP wave functions.

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- Note 1: $\hat{\psi}_{(X, Z)}(t, z)$

- Note 2: This answered some open questions, but left me with other. What does KP have to do with the Hermite product? What about all of the other KP wave functions? Let me talk.

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In other words, generating functions for XHermite functions are KP wave functions.

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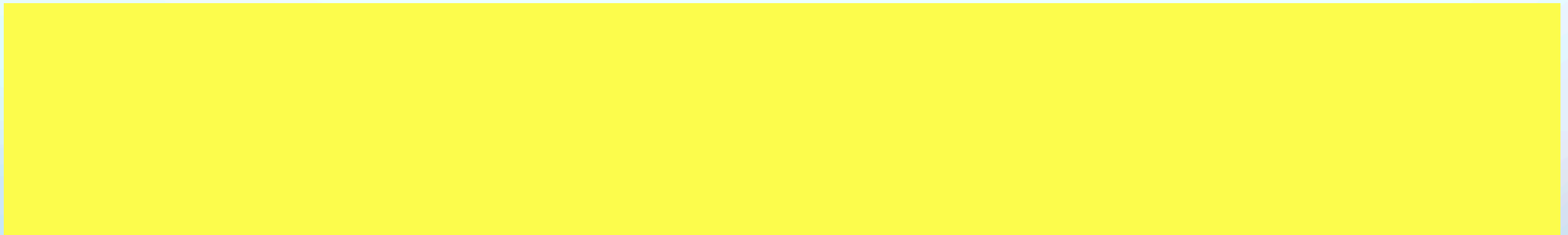
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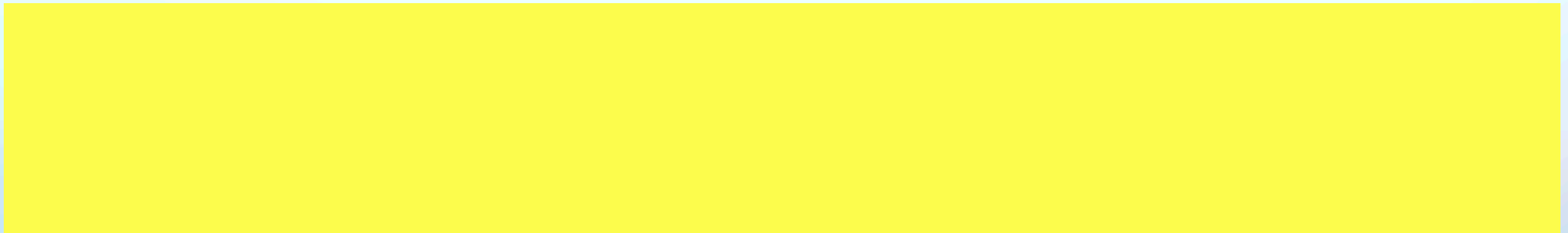
Step 1 of 3: Orthogonality via Generating Functions



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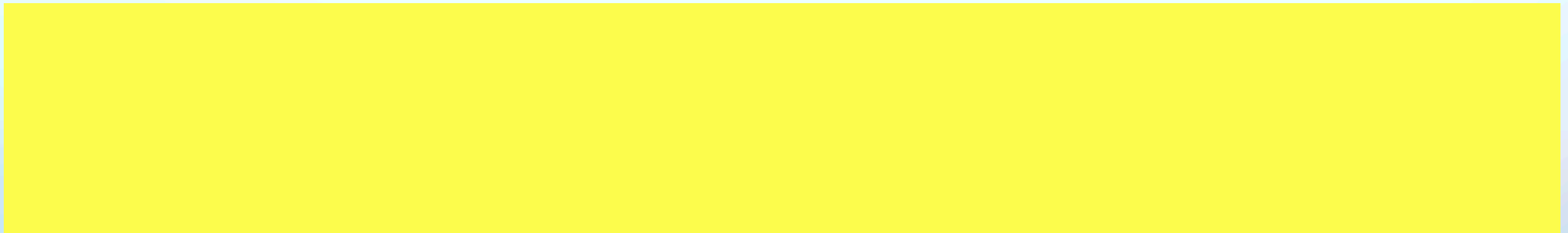
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- Lemma: The coefficients in the generating function $g(x, z)$ are orthogonal iff $\langle g(x, w), g(x, z) \rangle = \alpha(wz)$ is a function of the product “ wz ”.

- So $\int_{-\infty}^{\infty} \hat{\psi}_{(X_\lambda, Z_\lambda)}(x, -1/2, w) \hat{\psi}_{(X_\lambda, Z_\lambda)}(x, -1/2, z) e^{-x^2/2} dx = \alpha(wz)$. Why?

Step 2 of 3: Integral Formulation of KP



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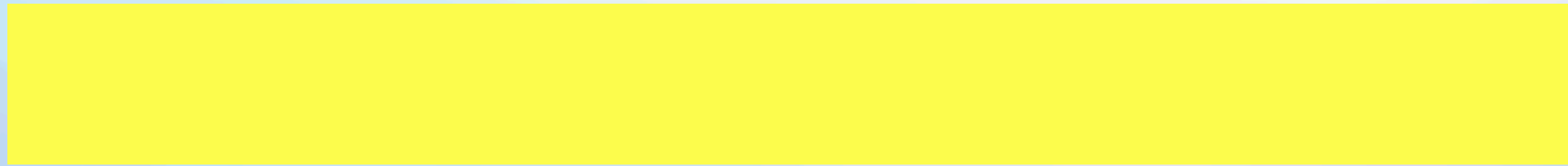
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- Consider these two differences: (a) one is an integral wrt z and the other wrt x (b) one has an “ $e^{-x^2/2}$ ” in it and other doesn’t.
- It was my intention to address (a) using the bispectral involution and worry about (b) afterwards.

Step 3 of 3: Using the Bispectral Involution



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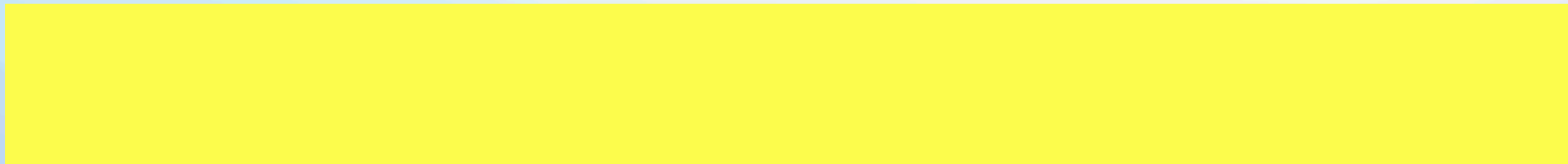
- If you apply the bispectral involution when $t_1 = x$, $t_j = 0$, $j \geq 2$

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- But, *miraculously* if $t_1 = x$, $t_2 = -1/2$, $t_j = 0$, $j \geq 3$

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Hermite weight!

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$$\oint \psi_{(X,Z)}(x,0,0,\dots,w) \psi_{(X^\star,Z^\star)}(x,0,0,z) dx = 0.$$

Where $(X^\star, Z^\star)$ has some messy formula

- But, *miraculously* if $t_1 = x$, $t_2 = -1/2$, $t_j = 0$, $j \geq 3$

$$\oint \psi_{(X,Z)}(x, -1/2, 0, \dots, z) \psi_{(-X^\top, Z^\top)}(x', -1/2, 0, z) dz = 0$$

is the same as

$$\oint \psi_{(X,Z)}(x, -1/2, 0, \dots, w) \psi_{(Z^\top, X^\top)}(x, -1/2, 0, z) e^{-x^2/2} dx = 0.$$

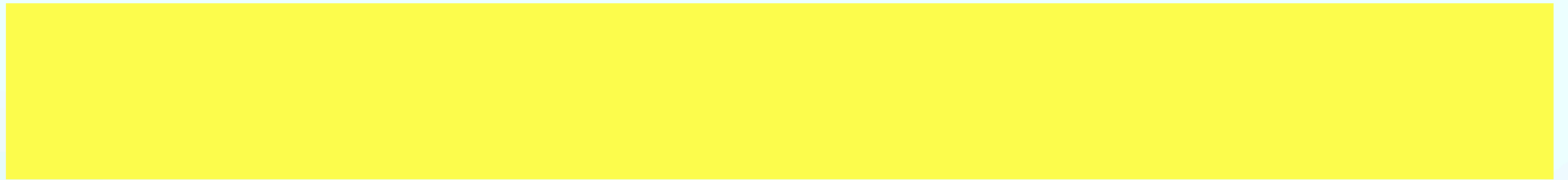
True for every $(X, Z) \in CM_1^n$. (On the right track!)

bispectral involution!

Hermite weight!

Key Lemma:

K-Milson-Gekhtman SIGMA 2026

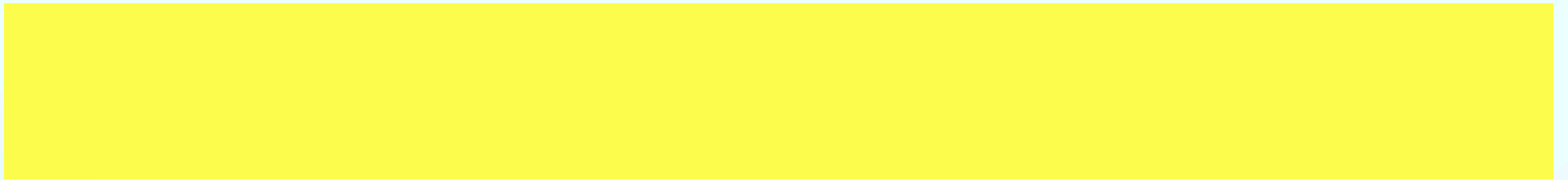


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- Let $(X, Z) \in CM_r^n$. Then we have a nice formula for

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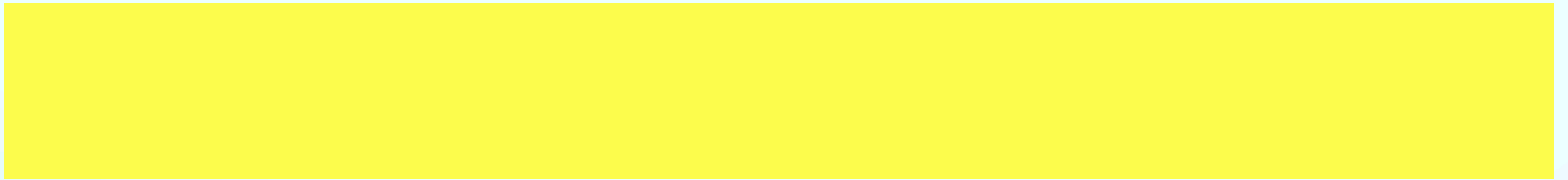
But, use $[X, Z] - I = ba^\top$ to move all of X 's to the right of all Z 's, and get something for which you can just read off an AD. Involves matrix mult, exp, and erf.

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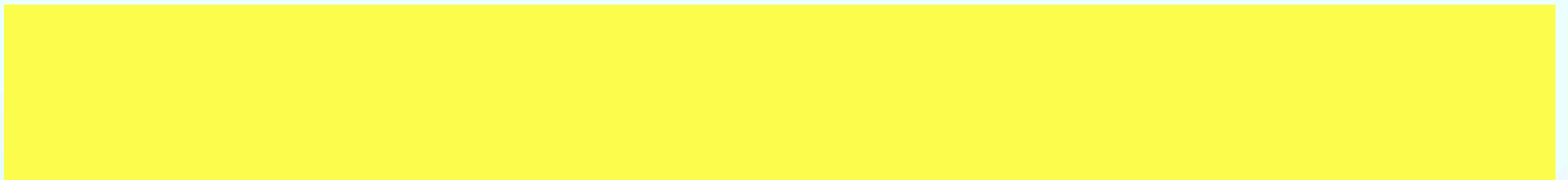
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There are poles for $x \in \mathbb{R}$. But, we don't need to worry about that because all of the residues are zero. (Cf. Haese-Hill/Hallnäs/Veselov, *Lett.Math.Phys.* 2016).

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- BTW This follows from a surprising fact: One can choose the roots of $\tau_{(X, Z)}(x, 0, 0, \dots)$ and $\tau_{(Z^\top, X^\top)}(x, 0, 0, \dots)$ independently. But $\tau_{(X, Z)}(x, -1/2, 0, \dots) = \tau_{(Z^\top, X^\top)}(x, -1/2, 0, \dots)$ is always true!

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K-Milson-Gekhtman SIGMA 2026



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So, we get a lot of Hermite orthogonality from KP Wave Functions and Bispectrality!

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- **Exceptional Hermites Rediscovered:** If $r = 1$, the latter holds exactly when $(X, Z) = (X_\lambda, Z_\lambda)$ for some partition.

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- There is also a link to Exceptional Krall polynomials (see forthcoming K-Milson in *J.Approx.Th.*) But no time for that now.