Lagrangian one-forms, gauge theories and applications

Vincent Caudrelier

UNIVERSITY OF LEEDS

Integrable systems: regularity, non-commutativity and random matrix theory
ICMS, Edinburgh, 15 - 19 June 2026

Based on various works with M. Dell'Atti, D. Harland, A.A. Singh, B. Vicedo

Plan

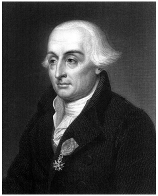
1. General context
2. Construction of Lagrangian one-forms
3. Venturing into non-commutativity
4. Lagrangian multiforms and gauge theories
5. Perspectives and conclusions

Plan

1. General context
2. Construction of Lagrangian one-forms
3. Venturing into non-commutativity
4. Lagrangian multiforms and gauge theories
5. Perspectives and conclusions

Notations: Lagrangians $\mathscr{L}$ vs Lax matrix L.

Basic motivation



Joseph-Louis

Lagrange

(1736-1813)

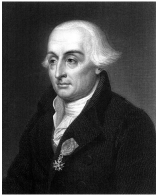
William Rowan Hamilton

(1805-65)

• Hamilton and/or Lagrange? Each formalism has pros and cons and complement each other.

1. General context

Basic motivation



Joseph-Louis

Lagrange

(1736-1813)

William Rowan Hamilton

(1805-65)

• Hamilton and/or Lagrange? Each formalism has pros and cons and complement each other.

But situation in integrable systems is quite unbalanced

→ Original purpose of Lagrangian multiforms theory: propose a variational formulation of integrability (Has evolved a bit since to something larger than just integrable.)

Lagrangian multiform theory in a nutshell (cf Mats's talk)

1) Theoretical framework part

• Introduce objects that generalise traditional Lagrangians and actions to capture the presence of simultaneous flows.

• Formulate a corresponding generalised variational principle to capture the mutual consistency of these flows/multidimensional consistency

• Derive the corresponding multitime Euler-Lagrange equations and the closure relation by applying the generalised variational principle

• Establish the relevant equations as a new/alternative integrability criterion

Lagrangian multiform theory in a nutshell (continued)

2) Explicit construction part

• Systematic construction methods

• Relation to other well established definition of integrability or other areas of mathematical physics (e.g. gauge theory, gauge symmetries, etc.), with possibly new insights (e.g. Hitchin system is variational)
$\rightarrow$ This is the focus of this (overview) talk

Lagrangian multiform theory in a nutshell (continued)

2) Explicit construction part

• Systematic construction methods

• Relation to other well established definition of integrability or other areas of mathematical physics (e.g. gauge theory, gauge symmetries, etc.), with possibly new insights (e.g. Hitchin system is variational)
$\rightarrow$ This is the focus of this (overview) talk

• What other well established integrability framework to compare to and to learn from?

• What can we do once this has been understood?

$\rightarrow$ These two questions are the drivers for the structure of this talk.

NB: Focus on 1-forms: systems of ODEs.

2. Construction of Lagrangian one-forms

- What systems do we have in mind
- What integrability criteria are we aiming to compare to
- What formalism to use?

- What systems do we have in mind
- What integrability criteria are we aiming to compare to
- What formalism to use?

- Back to “basics”: Liouville integrable systems (Hamiltonian)

- Phase space with (canonical) coordinates (q^μ, p_μ), $\mu = 1, \ldots, N$

- Collection of N independent functions $H_k(q^\mu, p_\mu)$ in involution

$$\{H_i, H_j\} = 0\,.$$

2. Construction of Lagrangian one-forms

- What systems do we have in mind
- What integrability criteria are we aiming to compare to
- What formalism to use?

- Back to "basics": Liouville integrable systems (Hamiltonian)

- Phase space with (canonical) coordinates (q^μ, p_μ), $\mu = 1, \dots, N$

- Collection of N independent functions $H_k(q^\mu, p_\mu)$ in involution

$$\{H_i, H_j\} = 0\,.$$

Integrable hierarchy of compatible Hamilton equations

$$\frac{dq^\mu}{dt^k} = \frac{\partial H_k}{\partial p_\mu}\,, \quad \frac{dp_\mu}{dt^k} = -\frac{\partial H_k}{\partial q^\mu}\,, \quad k = 1, \dots, N$$

$\to$ (local) existence of solutions $\mathbf{q}(t_1, \dots, t_N)$, $\mathbf{p}(t_1, \dots, t_N)$ depending on "multitime" $\mathbf{t} = (t_1, \dots, t_N)$.

Simple observation (and very well known!)

Hamilton's equations are variational

Simple observation (and very well known!)

Hamilton's equations are variational

- With

$$S[q^\mu, p_\mu] = \int_a^b \left(p_\mu \frac{dq^\mu}{dt} - H(q^\mu, p_\mu) \right) dt$$

we have

$$\delta_{q,p} S = 0 \Rightarrow \frac{dq^\mu}{dt} = \frac{\partial H}{\partial p_\mu}\,, \quad \frac{dp_\mu}{dt} = -\frac{\partial H}{\partial q^\mu}\,.$$

Simple observation (and very well known!)

Hamilton's equations are variational

- With

$$S[q^\mu, p_\mu] = \int_a^b \left(p_\mu \frac{dq^\mu}{dt} - H(q^\mu, p_\mu) \right) dt$$

we have

$$\delta_{q,p} S = 0 \Rightarrow \frac{dq^\mu}{dt} = \frac{\partial H}{\partial p_\mu}\,, \quad \frac{dp_\mu}{dt} = -\frac{\partial H}{\partial q^\mu}\,.$$

- Looking good for a *system* of Hamilton equations

$$\frac{dq^\mu}{dt^k} = \frac{\partial H_k}{\partial p_\mu}\,, \quad \frac{dp_\mu}{dt^k} = -\frac{\partial H_k}{\partial q^\mu}\,, \quad k = 1, \dots, N$$

but not completely trivial...

- Introduce the **phase-space Lagrangian one-form** (summation convention)

$$\mathscr{L} = p_\mu dq^\mu - H_i(\mathbf{p},\mathbf{q})\, dt^i$$

on $T^*M \times \mathbb{R}^N$.

[Caudrelier, Harland; Lett Math Phys '25]

- Introduce the **phase-space Lagrangian one-form** (summation convention)

$$\mathscr{L} = p_\mu dq^\mu - H_i(\mathbf{p}, \mathbf{q})\, dt^i$$

on $T^*M \times \mathbb{R}^N$.

[Caudrelier, Harland; Lett Math Phys '25]

- For any immersion on multitime space

$$\Sigma : \mathbb{R}^N \to T^*M \times \mathbb{R}^N\,, \quad (u^j) \mapsto (p_\mu(u), q^\mu(u), t^i(u))$$

consider the family of actions

$$\mathcal{S}_\Gamma[\Sigma] = \int_0^1 \left(p_\mu \frac{\partial q^\mu}{\partial u^j} - H_i(\mathbf{p}, \mathbf{q}) \frac{\partial t^i}{\partial u^j} \right) \frac{du^j}{ds}\, ds\,.$$

indexed by curves $\Gamma : (0,1) \to \mathbb{R}^N$, $s \mapsto (u^j(s))$.

Univariational principle and equations of motions

Theorem

The immersion Σ is a critical point of $\mathcal{S}_\Gamma[\Sigma]$ for all curves Γ iff the following equations of motion holds

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial H_i}{\partial p_\mu}\,, \quad \frac{\partial q^\mu}{\partial t^i} = -\frac{\partial H_i}{\partial q^\mu}\,, \quad \{H_i, H_j\} = 0$$

Univariational principle and equations of motions

Theorem

The immersion Σ is a critical point of $\mathcal{S}_\Gamma[\Sigma]$ for all curves Γ iff the following equations of motion holds

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial H_i}{\partial p_\mu}\,, \quad \frac{\partial q^\mu}{\partial t^i} = -\frac{\partial H_i}{\partial q^\mu}\,, \quad \{H_i, H_j\} = 0$$

• Notice not only the set of Hamilton eqs are derived variationally but also the involutivity of the functions H_j.

Univariational principle and equations of motions

Theorem

The immersion Σ is a critical point of $\mathcal{S}_\Gamma[\Sigma]$ for all curves Γ iff the following equations of motion holds

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial H_i}{\partial p_\mu}, \quad \frac{\partial q^\mu}{\partial t^i} = -\frac{\partial H_i}{\partial q^\mu}, \quad \{H_i, H_j\} = 0$$

• Notice not only the set of Hamilton eqs are derived variationally but also the involutivity of the functions H_j.
$\rightarrow$ Poisson involutivity is variational! More on this later in larger context.

Univariational principle and equations of motions

Theorem

The immersion Σ is a critical point of $\mathcal{S}_\Gamma[\Sigma]$ for all curves Γ iff the following equations of motion holds

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial H_i}{\partial p_\mu}\,, \quad \frac{\partial q^\mu}{\partial t^i} = -\frac{\partial H_i}{\partial q^\mu}\,, \quad \{H_i, H_j\} = 0$$

• Notice not only the set of Hamilton eqs are derived variationally but also the involutivity of the functions H_j.
$\rightarrow$ Poisson involutivity is variational! More on this later in larger context.

• OK nothing really constructive so far, "just" the important result that a **Liouville-Arnold integrable hierarchy** can be formulated variationally.
Let's keep thinking...

2. Construction of Lagrangian one-forms

• Another fundamental and well recognised integrability criterion: **Lax integrability**.

• A system of ODEs is Lax integrable if it can be cast into a Lax equation

$$\frac{dL}{dt} = [M(L), L]$$

• Another fundamental and well recognised integrability criterion: **Lax integrability**.

• A system of ODEs is Lax integrable if it can be cast into a Lax equation

$$\frac{dL}{dt} = [M(L), L]$$

• How to understand this? (open) Toda example (cf Jacobi matrix)

2. Construction of Lagrangian one-forms

- Another fundamental and well recognised integrability criterion: **Lax integrability**.

- A system of ODEs is Lax integrable if it can be cast into a Lax equation

$$\frac{dL}{dt} = [M(L), L]$$

- How to understand this? (open) Toda example (cf Jacobi matrix)

- Immediate consequence: $\mathrm{Tr}(L^k)$ is conserved $\frac{d}{dt}\mathrm{Tr}(L^k) = 0$.

2. Construction of Lagrangian one-forms

- Another fundamental and well recognised integrability criterion: **Lax integrability**.

- A system of ODEs is Lax integrable if it can be cast into a Lax equation

$$\frac{dL}{dt} = [M(L), L]$$

- How to understand this? (open) Toda example (cf Jacobi matrix)

- Immediate consequence: $\mathrm{Tr}(L^k)$ is conserved $\frac{d}{dt}\mathrm{Tr}(L^k) = 0$.

- Extension to **hierarchy of compatible Lax equations** for a Lax matrix L

$$\frac{dL}{dt^k} = [M_k, L]\,, \quad k = 1, \dots, N\,, \quad \frac{dM_j}{dt^k} - \frac{dM_k}{dt^j} + [M_k, M_j] = 0$$

Fundamental breakthrough

• Hamiltonian and Lax pair integrability intimately related.
Unification via classical r-matrix.

[Sklyanin, Semenov-Tian-Shansky, Reyman, Drinfel'd, Babelon-Viallet, Maillet...]

Fundamental breakthrough

- Hamiltonian and Lax pair integrability intimately related.

Unification via classical r-matrix.

[Sklyanin, Semenov-Tian-Shansky, Reyman, Drinfel'd, Babelon-Viallet, Maillet...]

- In short

$$\frac{dL}{dt^k} = \{H_k, L\}_R \Leftrightarrow \frac{dL}{dt^k} = [M_k, L]$$

Fundamental breakthrough

• Hamiltonian and Lax pair integrability intimately related.
Unification via classical r-matrix.

[Sklyanin, Semenov-Tian-Shansky, Reyman, Drinfel'd, Babelon-Viallet, Maillet...]

• In short

$$\frac{dL}{dt^k} = \{H_k, L\}_R \Leftrightarrow \frac{dL}{dt^k} = [M_k, L]$$

• Not only a beautiful idea but leads to constructive methods for integrable hierarchies!

Fundamental breakthrough

• Hamiltonian and Lax pair integrability intimately related.
Unification via classical r-matrix.

[Sklyanin, Semenov-Tian-Shansky, Reyman, Drinfel'd, Babelon-Viallet, Maillet...]

• In short

$$\frac{dL}{dt^k} = \{H_k, L\}_R \Leftrightarrow \frac{dL}{dt^k} = [M_k, L]$$

• Not only a beautiful idea but leads to constructive methods for integrable hierarchies!

• For our purposes: one easy and one more challenging aspect

Fundamental breakthrough

• Hamiltonian and Lax pair integrability intimately related.
Unification via classical r-matrix.

[Sklyanin, Semenov-Tian-Shansky, Reyman, Drinfel'd, Babelon-Viallet, Maillet...]

• In short

$$\frac{dL}{dt^k} = \{H_k, L\}_R \Leftrightarrow \frac{dL}{dt^k} = [M_k, L]$$

• Not only a beautiful idea but leads to constructive methods for integrable hierarchies!

• For our purposes: one easy and one more challenging aspect

Easy: natural Hamiltonians $H_k = \mathrm{Tr}(L^k)$ for our one-form;

2. Construction of Lagrangian one-forms

Fundamental breakthrough

• Hamiltonian and Lax pair integrability intimately related.
Unification via classical r-matrix.

[Sklyanin, Semenov-Tian-Shansky, Reyman, Drinfel'd, Babelon-Viallet, Maillet...]

• In short

$$\frac{dL}{dt^k} = \{H_k, L\}_R \Leftrightarrow \frac{dL}{dt^k} = [M_k, L]$$

• Not only a beautiful idea but leads to constructive methods for integrable hierarchies!

• For our purposes: one easy and one more challenging aspect

Easy: natural Hamiltonians $H_k = \mathrm{Tr}(L^k)$ for our one-form;

Not easy: what is the correct kinetic part '$p_\mu dq^\mu$' when working with Lax matrix L and Poisson bracket $\{\ ,\ \}_R$?

Technical tool: Semenov-Tian-Shansky's **Lie dialgebras**.

Technical tool: Semenov-Tian-Shansky's **Lie dialgebras**.

Key ingredients:

• A (matrix) Lie algebra $\mathfrak{g}$;

• A linear map $R : \mathfrak{g} \to \mathfrak{g}$ **solution of the modified Classical Yang-Baxter Equation**

$$[R(X), R(Y)] - R\left([R(X), Y] + [X, R(Y)]\right) = -[X, Y]\,,\ \forall\, X, Y \in \mathfrak{g}$$

Outputs:

- A second Lie bracket on $\mathfrak{g}$

$$[X,Y]_R = \frac{1}{2}\left([R(X),Y] + [X,R(Y)]\right)$$

Outputs:

- A second Lie bracket on $\mathfrak{g}$

$$[X,Y]_R = \frac{1}{2}\left([R(X),Y] + [X,R(Y)]\right)$$

$\rightarrow$ Denote g_R the corresponding Lie algebra, G_R its Lie group and $\cdot_R$ the multiplication in G_R.

Outputs:

- A second Lie bracket on $\mathfrak{g}$

$$[X, Y]_R = \frac{1}{2}\left([R(X), Y] + [X, R(Y)]\right)$$

$\rightarrow$ Denote g_R the corresponding Lie algebra, G_R its Lie group and $\cdot_R$ the multiplication in G_R.

- A Poisson bracket $\{\ ,\ \}_R$ of Lie-Poisson type. Natural phase spaces are symplectic leaves of this Poisson bracket.

Outputs:

- A second Lie bracket on $\mathfrak{g}$

$$[X, Y]_R = \frac{1}{2}\left([R(X), Y] + [X, R(Y)]\right)$$

$\rightarrow$ Denote g_R the corresponding Lie algebra, G_R its Lie group and $\cdot_R$ the multiplication in G_R.

- A Poisson bracket $\{\ ,\ \}_R$ of Lie-Poisson type. Natural phase spaces are symplectic leaves of this Poisson bracket.

$\rightarrow$ Tells us how to parametrise L and what to use for the kinetic part!

Outputs:

- A second Lie bracket on $\mathfrak{g}$

$$[X, Y]_R = \frac{1}{2}\left([R(X), Y] + [X, R(Y)]\right)$$

$\rightarrow$ Denote g_R the corresponding Lie algebra, G_R its Lie group and $\cdot_R$ the multiplication in G_R.

- A Poisson bracket $\{\ ,\ \}_R$ of Lie-Poisson type. Natural phase spaces are symplectic leaves of this Poisson bracket.

$\rightarrow$ Tells us how to parametrise L and what to use for the kinetic part!

- Semenov-Tian-Shansky's theorem: Ad**-invariant functions** H_k give a set of commuting Hamiltonian flows in Lax form:

$$\partial_{t_k} L = \{L, H_k\}_R = [M_k, L]\,, \quad M_k = \frac{1}{2} R\nabla H_k(L)\,,\ \{H_j, H_k\}_R = 0\,.$$

Outputs:

- A second Lie bracket on $\mathfrak{g}$

$$[X,Y]_R = \frac{1}{2}\left([R(X),Y] + [X,R(Y)]\right)$$

$\rightarrow$ Denote g_R the corresponding Lie algebra, G_R its Lie group and $\cdot_R$ the multiplication in G_R.

- A Poisson bracket $\{\ ,\ \}_R$ of Lie-Poisson type. Natural phase spaces are symplectic leaves of this Poisson bracket.

$\rightarrow$ Tells us how to parametrise L and what to use for the kinetic part!

- Semenov-Tian-Shansky's theorem: Ad**-invariant functions** H_k give a set of commuting Hamiltonian flows in Lax form:

$$\partial_{t_k} L = \{L, H_k\}_R = [M_k, L]\,, \quad M_k = \frac{1}{2} R \nabla H_k(L)\,,\ \{H_j, H_k\}_R = 0\,.$$

$\rightarrow$ Tells us what to choose for $H_k(L)$.

2. Construction of Lagrangian one-forms

With all this, here is our answer **[Caudrelier, Dell'Atti, Singh; Lett Math Phys '23]**

2. Construction of Lagrangian one-forms

With all this, here is our answer **[Caudrelier, Dell'Atti, Singh; Lett Math Phys '23]**

• Choose $H_k(L) = \mathrm{Tr}(L^k)$. Parametrise $L = \varphi \cdot_R \Lambda \cdot_R \varphi^{-1}$
φ: group-valued **containing the dynamical degrees of freedom**
Λ: algebra-valued non-dynamical **constant matrix**.

2. Construction of Lagrangian one-forms

With all this, here is our answer **[Caudrelier, Dell'Atti, Singh; Lett Math Phys '23]**

- Choose $H_k(L) = \mathrm{Tr}(L^k)$. Parametrise $L = \varphi \cdot_R \Lambda \cdot_R \varphi^{-1}$

φ: group-valued **containing the dynamical degrees of freedom**

Λ: algebra-valued non-dynamical **constant matrix**.

- Define $\mathscr{L} = \mathrm{Tr}\left(L\,,\, d\varphi \cdot_R \varphi^{-1}\right) - H_k(L)\, dt^k$

2. Construction of Lagrangian one-forms

With all this, here is our answer **[Caudrelier, Dell'Atti, Singh; Lett Math Phys '23]**

- Choose $H_k(L) = \mathrm{Tr}(L^k)$. Parametrise $L = \varphi \cdot_R \Lambda \cdot_R \varphi^{-1}$

φ: group-valued **containing the dynamical degrees of freedom**

Λ: algebra-valued non-dynamical **constant matrix**.

- Define $\mathscr{L} = \mathrm{Tr}\left(L \,,\, d\varphi \cdot_R \varphi^{-1}\right) - H_k(L)\, dt^k$

Theorem

The univariational principle applied to $S = \int_\Gamma \mathscr{L}$ yields

$$\partial_{t_k} L = \frac{1}{2}[R\nabla H_k(L), L]\,, \qquad k = 1, \dots, N\,, \tag{1}$$

as its set of compatible Hamilton equations, in Lax form. The closure relation holds by design since on solutions of (1) we have

$$d\mathscr{L} = 0 \Leftrightarrow \{H_k, H_\ell\}_R = 0\,.$$

• By the generality of the construction, this applies to any integrable system fitting the Lie dialgebra framework (a lot of them)

• "Famous" examples: open and periodic Toda chains, rational Gaudin models (and cyclotomic versions),

• Purely variational description of these models and validation of Lagrangian multiform theory as an alternative integrability criterion (cf Lax integrability and CYBE integrability definitions).

- Back to generic **phase-space Lagrangian one-form**

$$\mathscr{L} = p_\mu dq^\mu - H_i(\mathbf{p}, \mathbf{q})\, dt^i$$

on $T^*M \times \mathbb{R}^N$ and the univariational principle applied to the family of actions $\mathcal{S}_\Gamma[\Sigma]$.

- Recall it delivered the following equations:

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial H_i}{\partial p_\mu}\,, \quad \frac{\partial q^\mu}{\partial t^i} = -\frac{\partial H_i}{\partial q^\mu}\,, \quad \{H_i, H_j\} = 0$$

Question: what if we want to extend to $\{H_i, H_j\} = c_{ij}^k H_k$ i.e. a case where the Hamiltonian vector fields X_{H_i} no longer generate commuting flows but flow that close on a Lie algebra?

- Examine $\mathscr{L}$ again, in particular the potential term $H_i(\mathbf{p}, \mathbf{q})\, dt^i$.

- **Key observation:** this is the pairing of a $\mathfrak{g}^*$-valued function $H : T^*M \to \mathfrak{g}^*$ with the Maurer-Cartan form $g^{-1}dg$ in the case of an abelian Lie group, with coordinates t^i:

$$H_i(\mathbf{p}, \mathbf{q})\, dt^i = \langle H, g^{-1}dg \rangle \,.$$

$\to$ allows for immediate generalisation to a non abelian Lie group G.

A "provocative" statement: *Hamiltonian* Lie group actions on a manifold are *Lagrangian*

A "provocative" statement: *Hamiltonian* Lie group actions on a manifold are *Lagrangian*

- Introduce

$$\mathscr{L} = p_\mu dq^\mu - \langle H, g^{-1}dg \rangle$$

on $T^*M \times G$.

A "provocative" statement: *Hamiltonian* Lie group actions on a manifold are *Lagrangian*

• Introduce

$$\mathscr{L} = p_\mu dq^\mu - \langle H, g^{-1}dg \rangle$$

on $T^*M \times G$.

• With local coordinates on G, say τ^k, $\{E_i\}$ a basis for $\mathfrak{g}$, note that $g^{-1}dg = E_i \otimes \theta^i$ and $\theta^i = \theta^i_k d\tau^k$ so that

$$\langle H, g^{-1}dg \rangle = H_i \theta^i = H_i \theta^i_k d\tau^k \,, \quad H_i = (H, E_i)\,.$$

A "provocative" statement: *Hamiltonian* Lie group actions on a manifold are *Lagrangian*

• Introduce

$$\mathscr{L} = p_\mu dq^\mu - \langle H, g^{-1}dg \rangle$$

on $T^*M \times G$.

• With local coordinates on G, say τ^k, $\{E_i\}$ a basis for $\mathfrak{g}$, note that $g^{-1}dg = E_i \otimes \theta^i$ and $\theta^i = \theta^i_k d\tau^k$ so that

$$\langle H, g^{-1}dg \rangle = H_i \theta^i = H_i \theta^i_k d\tau^k \,, \quad H_i = (H, E_i) \,.$$

• Apply the univariational principle to get

$$\frac{dq^\mu}{d\tau^k} = \frac{\partial H_j}{\partial p_\mu} \theta^j_k \,,$$

$$\frac{dp_\mu}{d\tau^k} = -\frac{\partial H_j}{\partial q^\mu} \theta^j_k \,,$$

$$\{H_i, H_j\} = c^k_{ij} H_k \,,$$

where c^k_{ij} are the structure constants of $\mathfrak{g}$.

Interpretation and clarification of the “provocative” statement

Interpretation and clarification of the "provocative" statement

$$\frac{dq^\mu}{d\tau^k} = \frac{\partial H_j}{\partial p_\mu}\theta^j_k\,, \tag{2}$$

$$\frac{dp_\mu}{d\tau^k} = -\frac{\partial H_j}{\partial q^\mu}\theta^j_k\,, \tag{3}$$

$$\{H_i, H_j\} = c^k_{ij} H_k\,, \tag{4}$$

- (2)-(3) tell us that the fundamental vector fields of the (cotangent lift) action of G on T^*M are Hamiltonian
- (4) tells that H is the moment map of this action satisfying the required (infinitesimal) equivariance property.

Interpretation and clarification of the “provocative” statement

$$\frac{dq^\mu}{d\tau^k} = \frac{\partial H_j}{\partial p_\mu}\theta^j_k\,, \tag{2}$$

$$\frac{dp_\mu}{d\tau^k} = -\frac{\partial H_j}{\partial q^\mu}\theta^j_k\,, \tag{3}$$

$$\{H_i, H_j\} = c^k_{ij}H_k\,, \tag{4}$$

• (2)-(3) tell us that the fundamental vector fields of the (cotangent lift) action of G on T^*M are Hamiltonian

• (4) tells that H is the moment map of this action satisfying the required (infinitesimal) equivariance property.

→ Hamiltonian Lie group actions are variational! Lagrangian multiform theory necessary for this result.

Interpretation and clarification of the "provocative" statement

$$\frac{dq^\mu}{d\tau^k} = \frac{\partial H_j}{\partial p_\mu}\theta^j_k\,, \tag{2}$$

$$\frac{dp_\mu}{d\tau^k} = -\frac{\partial H_j}{\partial q^\mu}\theta^j_k\,, \tag{3}$$

$$\{H_i, H_j\} = c^k_{ij} H_k\,, \tag{4}$$

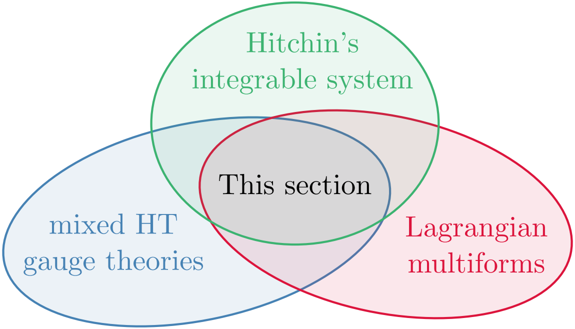
• (2)-(3) tell us that the fundamental vector fields of the (cotangent lift) action of G on T^*M are Hamiltonian

• (4) tells that H is the moment map of this action satisfying the required (infinitesimal) equivariance property.

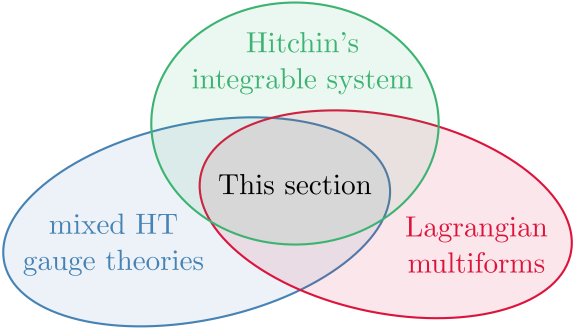
→ Hamiltonian Lie group actions are variational! Lagrangian multiform theory necessary for this result.

• Commutative case $c^k_{ij} = 0$ recovers case of commuting flows and geometric interpretation of combined Hamiltonian flows as action of abelian group $G = \mathbb{R}^n$.

4. Lagrangian multiforms and gauge theories



[Caudrelier, Harland, Singh, Vicedo; Comm Math Phys '26]



[Caudrelier, Harland, Singh, Vicedo; Comm Math Phys '26]

Why?

- Emergence of two independent variational approaches to integrable systems:

• Emergence of two independent variational approaches to integrable systems:

1. **Lagrangian multiforms**

[Lobb, Nijhoff '09]

$\rightarrow$ variational description of an integrable hierarchy with a generalised action and action principle.

4. Lagrangian multiforms and gauge theories

• Emergence of two independent variational approaches to integrable systems:

1. **Lagrangian multiforms**

[Lobb, Nijhoff '09]

$\rightarrow$ variational description of an integrable hierarchy with a generalised action and action principle.

2. **Mixed Holomorphic-topological gauge theories**, e.g. 4d Chern-Simons theory: **[Costello, Witten, Yamazaki, '13-18]**
or 3d BF theory **[Vicedo, Winstone, '22]**
$\rightarrow$ gauge theoretic origin of a single integrable model with a traditional action of a gauge theory but on enlarged "spacetime" (typically $M \times \mathbb{C}P^1$).

4. Lagrangian multiforms and gauge theories

• Emergence of two independent variational approaches to integrable systems:

1. **Lagrangian multiforms**

[Lobb, Nijhoff '09]

→ variational description of an integrable hierarchy with a generalised action and action principle.

2. **Mixed Holomorphic-topological gauge theories**, e.g. 4d Chern-Simons theory: **[Costello, Witten, Yamazaki, '13-18]**
or 3d BF theory **[Vicedo, Winstone, '22]**
→ gauge theoretic origin of a single integrable model with a traditional action of a gauge theory but on enlarged "spacetime" (typically $M \times \mathbb{C}P^1$).

• **Hitchin's integrable system**: remarkable integrable system sitting naturally across 1) and 2) because
- natural relation to gauge theory (via its definition)
- its variational formulation requires a multiform as no preferred "physical" Hamiltonian

Gauged Lagrangian one-forms

• General philosophy: many integrable systems arise as symplectic quotients of "easier Hamiltonian systems with symmetries on bigger space".

Gauged Lagrangian one-forms

• General philosophy: many integrable systems arise as symplectic quotients of “easier Hamiltonian systems with symmetries on bigger space”.

• Let G act on T^*M by Hamiltonian action with moment map $\mu : T^*M \to \mathfrak{g}^*$ and $\{H_i, \mu\} = 0$ (invariant Hamiltonians). Reduced phase space of interest is typically $\mu^{-1}(0)/G$.

Gauged Lagrangian one-forms

• General philosophy: many integrable systems arise as symplectic quotients of "easier Hamiltonian systems with symmetries on bigger space".

• Let G act on T^*M by Hamiltonian action with moment map $\mu : T^*M \to \mathfrak{g}^*$ and $\{H_i, \mu\} = 0$ (invariant Hamiltonians). Reduced phase space of interest is typically $\mu^{-1}(0)/G$.

$\to$ How to implement this idea in the variational formalism of Lagrangian one-forms?

Gauged Lagrangian one-forms

• General philosophy: many integrable systems arise as symplectic quotients of “easier Hamiltonian systems with symmetries on bigger space”.

• Let G act on T^*M by Hamiltonian action with moment map $\mu : T^*M \to \mathfrak{g}^*$ and $\{H_i, \mu\} = 0$ (invariant Hamiltonians). Reduced phase space of interest is typically $\mu^{-1}(0)/G$.

→ How to implement this idea in the variational formalism of Lagrangian one-forms?

1. Enforce $\mu = 0$ using a **Lagrangian multiplier** $\mathcal{A} = \mathcal{A}_j du^j$, $\mathcal{A}_j = \mathcal{A}_j^a X_a \in \mathfrak{g}$

$$\mathcal{S}_\Gamma[\Sigma, \mathcal{A}] = \mathcal{S}_\Gamma[\Sigma] + \int_0^1 \Gamma^* \langle \Sigma^* \mu, \mathcal{A} \rangle$$

2. Pass to the quotient by **gauging the symmetry**: $\forall g \in C^\infty(\mathbb{R}^N, G)$: $\Sigma(u) \mapsto g(u) \cdot \Sigma(u)$, $\mathcal{A} \mapsto g\mathcal{A}g^{-1} - d_{\mathbb{R}^N} g g^{-1}$

Gauged equations of motion

$$\mu = 0\,, \tag{5}$$

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial}{\partial p_\mu}\left(H_i - \mu_a \widetilde{\mathcal{A}}_i^a\right)\,, \tag{6}$$

$$\frac{\partial p_\mu}{\partial t^i} = \frac{\partial}{\partial q^\mu}\left(H_i - \mu_a \widetilde{\mathcal{A}}_i^a\right)\,, \tag{7}$$

$$\{H_i, H_j\} = 0$$

• (5) restricts the dynamics to $\mu^{-1}(0) \subset T^*M$ and (6)-(7) restrict further to $\mu^{-1}(0)/G$ as desired.

• Crucially, if G action is free, (6)-(7) imply zero curvature equation for gauge field

$$\partial_{t_i}\widetilde{\mathcal{A}}_j - \partial_{t_j}\widetilde{\mathcal{A}}_i + \left[\widetilde{\mathcal{A}}_i, \widetilde{\mathcal{A}}_j\right] = 0$$

Application to Hitchin's system

• Fix a compact Riemann surface $\mathcal{C}$ of genus g (≥ 2 for now) and G a complex Lie group

• Introduce $\mathrm{Bun}_G(\mathcal{C})$: moduli space of isomorphism classes of stable holomorphic principal G-bundles over $\mathcal{C}$.

• First remarkable fact: $\mathrm{Bun}_G(\mathcal{C})$ has finite dimension $(g-1)\mathrm{dim}G$.

• Phase space of Hitchin system is $T^*\mathrm{Bun}_G(\mathcal{C})$.

• Second remarkable fact: $T^*\mathrm{Bun}_G(\mathcal{C})$ can be identified with a symplectic quotient $\mu^{-1}(0)/\mathcal{G}$ starting from appropriate $\mathcal{M}$ and group $\mathcal{G}$.

Description of "big phase space"

• Fix a smooth principle G-bundle $\mathcal{P} \to \mathcal{C}$. $\mathcal{P}$ is specified by smooth transition functions relative to a cover $\{U_I\}$ of $\mathcal{C}$: $g_{IJ} : U_I \cap U_J \to G$

Description of "big phase space"

- Fix a smooth principle G-bundle $\mathcal{P} \to \mathcal{C}$. $\mathcal{P}$ is specified by smooth transition functions relative to a cover $\{U_I\}$ of $\mathcal{C}$: $g_{IJ} : U_I \cap U_J \to G$
- Let $\mathcal{M}$ be the space of $\mathfrak{g}$-valued $(0,1)$ connections A'' on $\mathcal{P}$:

$$A''_{\bar{z}_I} d\bar{z}_I = g_{IJ} A''_{\bar{z}_J} d\bar{z}_J g_{IJ}^{-1} - \bar{\partial} g_{IJ} g_{IJ}^{-1} \quad \text{(analog of } \; q^\mu\text{)}$$

Description of "big phase space"

- Fix a smooth principle G-bundle $\mathcal{P} \to \mathcal{C}$. $\mathcal{P}$ is specified by smooth transition functions relative to a cover $\{U_I\}$ of $\mathcal{C}$: $g_{IJ} : U_I \cap U_J \to G$

- Let $\mathcal{M}$ be the space of $\mathfrak{g}$-valued $(0,1)$ connections A'' on $\mathcal{P}$:

$$A''_{\bar{z}_I} d\bar{z}_I = g_{IJ} A''_{\bar{z}_J} d\bar{z}_J g_{IJ}^{-1} - \bar{\partial} g_{IJ} g_{IJ}^{-1} \quad \text{(analog of } q^\mu)$$

- An element of $T^*_{A''}\mathcal{M}$ is a $\mathfrak{g}^*$-valued $(1,0)$-form B:

$$B_{z_I} dz_I = \mathrm{Ad}^*_{g_{IJ}} B_{z_J} dz_J \quad \text{(analog of } p_\mu)$$

Description of "big phase space"

- Fix a smooth principle G-bundle $\mathcal{P} \to \mathcal{C}$. $\mathcal{P}$ is specified by smooth transition functions relative to a cover $\{U_I\}$ of $\mathcal{C}$: $g_{IJ} : U_I \cap U_J \to G$
- Let $\mathcal{M}$ be the space of $\mathfrak{g}$-valued $(0,1)$ connections A'' on $\mathcal{P}$:

$$A''_{\bar{z}_I} d\bar{z}_I = g_{IJ} A''_{\bar{z}_J} d\bar{z}_J g_{IJ}^{-1} - \bar{\partial} g_{IJ} g_{IJ}^{-1} \quad \text{(analog of } \; q^\mu)$$

- An element of $T^*_{A''}\mathcal{M}$ is a $\mathfrak{g}^*$-valued $(1,0)$-form B:

$$B_{z_I} dz_I = \mathrm{Ad}^*_{g_{IJ}} B_{z_J} dz_J \quad \text{(analog of } \; p_\mu)$$

- Tautological symplectic form

$$\omega = \frac{1}{2i\pi} \int_{\mathcal{C}} \langle \delta B, \delta A'' \rangle \quad \text{(analog of } \; dp_\mu \wedge dq^\mu)$$

Invariant commuting Hamiltonians and dynamics

• Adapt Hitchin original idea to our case: Hamiltonians on $T^*\mathcal{M}$ built from G invariant homogeneous polynomials P_i evaluated on B at certain generic points $_i \in C$. In short,

$$H_i(B) = P_i(B_{z_I}(\mathsf{q}_i))\,, \quad i = 1, \dots, N = (g-1)\mathrm{dim}G\,.$$

Invariant commuting Hamiltonians and dynamics

• Adapt Hitchin original idea to our case: Hamiltonians on $T^*\mathcal{M}$ built from G invariant homogeneous polynomials P_i evaluated on B at certain generic points $_i \in C$. In short,

$$H_i(B) = P_i(B_{z_I}(\mathsf{q}_i))\,, \quad i = 1, \dots, N = (g-1)\mathrm{dim}G\,.$$

• We have the ingredients to form a phase space Lagrangian one-form on $T^*\mathcal{M} \times \mathbb{R}^N$

$$\mathscr{L}^0 = \frac{1}{2i\pi} \int_{\mathcal{C}} \langle B, \delta A'' \rangle - H_i(B) dt^i$$

But $T^*\mathcal{M}$ too big $\rightarrow$ reduce by a group action.

Group action

• Let $\mathcal{G}$ be the group of automorphisms of $\mathcal{P}$: an element in $\mathcal{G}$ is a family of maps $g_I : U_I \to G$ such that $g_I g_{IJ} = g_{IJ} g_J$ on $U_I \cap U_j \neq \emptyset$.

Group action

• Let $\mathcal{G}$ be the group of automorphisms of $\mathcal{P}$: an element in $\mathcal{G}$ is a family of maps $g_I : U_I \to G$ such that $g_I g_{IJ} = g_{IJ} g_J$ on $U_I \cap U_j \neq \emptyset$.

• $\mathcal{G}$ acts on $\mathcal{M}$ by gauge transformations

$$A'' \mapsto {}^g A'' = g\, A''\, g^{-1} - \bar{\partial} g\, g^{-1}$$

Group action

• Let $\mathcal{G}$ be the group of automorphisms of $\mathcal{P}$: an element in $\mathcal{G}$ is a family of maps $g_I : U_I \to G$ such that $g_I g_{IJ} = g_{IJ} g_J$ on $U_I \cap U_j \neq \emptyset$.

• $\mathcal{G}$ acts on $\mathcal{M}$ by gauge transformations

$$A'' \mapsto {}^g A'' = g\, A''\, g^{-1} - \bar{\partial} g\, g^{-1}$$

• Action lifts to $T^*\mathcal{M}$: $B \mapsto \mathrm{Ad}^*_g\, B$

$\to$ Hamiltonian action with moment map $\mu : T^*\mathcal{M} \to \mathrm{Lie}(\mathcal{G})^*$

$$\langle \mu, X \rangle = \frac{1}{2i\pi} \int_{\mathcal{C}} \langle B, \bar{\partial} X + [A'', X] \rangle \equiv \frac{1}{2i\pi} \int_{\mathcal{C}} \langle B, \bar{\partial}^{A''} X \rangle\,, \;\; X \in \mathrm{Lie}(\mathcal{G})\,.$$

Gauged Lagrangian one-form

Now we have all the ingredients for the gauged Lagrangian one-form and related action describing the Hitchin system on $\mu^{-1}(0)/\mathcal{G}$ following our procedure:

$$\begin{aligned}\mathcal{S}_\Gamma[\Sigma, \mathcal{A}] &= \int_0^1 \Gamma^* \left(\frac{1}{2i\pi} \int_{\mathcal{C}} \langle B, d_{\mathbb{R}^N} A'' \rangle - H_i(B) d_{\mathbb{R}^N} t^i \right) \\ &\quad + \int_0^1 \Gamma^* \langle \Sigma^* \mu, \mathcal{A} \rangle \\ &= \int_0^1 \Gamma^* \left(\frac{1}{2i\pi} \int_{\mathcal{C}} \langle B, d_{\mathbb{R}^N} A'' - \bar{\partial}^{A''} \mathcal{A} \rangle - H_i(B) d_{\mathbb{R}^N} t^i \right)\end{aligned}$$

Theorem

The gauged action for the Hitchin system takes the form of a 3d BF Lagrangian one-form with type B defects

$$\mathcal{S}_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \Gamma^* \left(\frac{1}{2i\pi} \int_{\mathcal{C}} \langle B, F_A \rangle - H_i(B) d_{\mathbb{R}^N} t^i \right)$$

F_A curvature of **assembled** connection
$A \equiv A'' + \mathcal{A} \sim A''_{\bar{z}_I} d\bar{z}_I + \mathcal{A}_j du^j$

Theorem

The gauged action for the Hitchin system takes the form of a 3d BF Lagrangian one-form with type B defects

$$\mathcal{S}_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \Gamma^* \left(\frac{1}{2i\pi} \int_{\mathcal{C}} \langle B, F_A \rangle - H_i(B) d_{\mathbb{R}^N} t^i \right)$$

F_A curvature of **assembled** connection
$A \equiv A'' + \mathcal{A} \sim A''_{\bar{z}_I} d\bar{z}_I + \mathcal{A}_j du^j$

- Equations of motion:

$$\partial_{\bar{z}_I} \widetilde{\mathcal{A}}^I_i - \partial_{t^i} A''_{\bar{z}_I} + [A''_{\bar{z}_I}, \widetilde{\mathcal{A}}^I_i] = 2\pi i \, \nabla P_i\big(B_{z_I}(\mathsf{q}_i)\big) \delta\big(z_I - z_I(\mathsf{q}_i)\big) \,,$$
$$\partial_{\bar{z}_I} B_{z_I} + \mathrm{ad}^*_{A''_{\bar{z}_I}} B_{z_I} = 0 \,, \quad \partial_{t^j} B_{z_I} + \mathrm{ad}^*_{\widetilde{\mathcal{A}}^I_j} B_{z_I} = 0 \,,$$
$$\partial_{t^i} \widetilde{\mathcal{A}}^I_j - \partial_{t^j} \widetilde{\mathcal{A}}^I_i + \big[\widetilde{\mathcal{A}}^I_i, \widetilde{\mathcal{A}}^I_j\big] = 0 \,.$$

Hitchin system in Lax form: holomorphic description

• Link to holomorphic description obtained by going to $A''_{\bar{z}_I} = 0$ and solving $\mu = 0$. This gives $\partial_{\bar{z}_I} B_{z_I} = 0 \Rightarrow B = B_{hol}$.

Hitchin system in Lax form: holomorphic description

• Link to holomorphic description obtained by going to $A''_{\bar{z}_I} = 0$ and solving $\mu = 0$. This gives $\partial_{\bar{z}_I} B_{z_I} = 0 \Rightarrow B = B_{hol}$.

• In this local trivialisation, set $L = B_{hol}$ and call it Lax "matrix". Equations of motion then read

$$\partial_{t^j} L_{z_I} + \mathrm{ad}^*_{\widetilde{\mathcal{A}}^I_j} L_{z_I} = 0\,,$$
$$\partial_{\bar{z}_I} \widetilde{\mathcal{A}}^I_i = 2\pi i\, \nabla P_i\big(L_{z_I}(\mathsf{q}_i)\big)\delta\big(z_I - z_I(\mathsf{q}_i)\big)\,,$$
$$\partial_{t^i} \widetilde{\mathcal{A}}^I_j - \partial_{t^j} \widetilde{\mathcal{A}}^I_i + \big[\widetilde{\mathcal{A}}^I_i, \widetilde{\mathcal{A}}^I_j\big] = 0\,.$$

Hitchin system in Lax form: holomorphic description

• Link to holomorphic description obtained by going to $A''_{\bar{z}_I}=0$ and solving $\mu=0$. This gives $\partial_{\bar{z}_I}B_{z_I}=0\Rightarrow B=B_{hol}$.

• In this local trivialisation, set $L=B_{hol}$ and call it Lax "matrix". Equations of motion then read

$$\begin{aligned}
&\partial_{t^j}L_{z_I}+\mathrm{ad}^*_{\widetilde{\mathcal{A}}^I_j}L_{z_I}=0\,,\\
&\partial_{\bar{z}_I}\widetilde{\mathcal{A}}^I_i=2\pi i\,\nabla P_i\big(L_{z_I}(\mathsf{q}_i)\big)\delta\big(z_I-z_I(\mathsf{q}_i)\big)\,,\\
&\partial_{t^i}\widetilde{\mathcal{A}}^I_j-\partial_{t^j}\widetilde{\mathcal{A}}^I_i+\big[\widetilde{\mathcal{A}}^I_i,\widetilde{\mathcal{A}}^I_j\big]=0\,.
\end{aligned}$$

• Identify $\mathfrak{g}^*$ and $\mathfrak{g}$, set $M^I_i=-\widetilde{\mathcal{A}}^I_i=\frac{\nabla P_i\big(L_{z_I}(\mathsf{q}_i)\big)}{z_I-z_I(\mathsf{q}_i)}+O(1)$

$\rightarrow$ **Hierarchy of compatible Lax equations!**

$$\partial_{t^i}L_{z_I}=[M^I_i,L_{z_I}]\,,\quad \partial_{t^i}M^I_j-\partial_{t^j}M^I_i-\big[M^I_i,M^I_j\big]=0\,.$$

Natural question: Can we complete the following commutative diagram?

$$\begin{tikzcd}
S_\Gamma[\Sigma, \mathcal{A}] \arrow[r, "\text{holomorphic}", "\text{description}"'] \arrow[d, "\text{univariational principle}"'] & \text{Unifying action} \arrow[d, "\text{univariational principle}"] \\
\text{Variational equations} \arrow[r, "\text{holomorphic}", "\text{description}"'] & \partial_{t^i} L = [M_i, L]
\end{tikzcd}$$

$\rightarrow$ Yes: leads to unifying action depending purely on holomorphic data: Lax matrix L and holomorphic transition function describing $[\mathcal{P}_{hol}]$

Unifying action in holomorphic description

- Fix $\mathsf{p} \in \mathcal{C}$. Define U_0 small neighbourhood of p and $U_1 = \mathcal{C} \setminus \{\mathsf{p}\}$
$\rightarrow$ cover of $\mathcal{C}$ which gives a trivialisation of $\mathcal{P}$ with transition function g_{01}.

- Moving to trivialisation $A_0'' = A_1'' = 0$ gives holomorphic transition function $\tilde{g}_{01} \equiv \gamma$.

Unifying action in holomorphic description

Theorem

The 3d mixed BF action

$$\mathcal{S}_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \Gamma^* \left(\frac{1}{2i\pi} \int_C \langle B, F_A \rangle - H_i(B) d_{\mathbb{R}^N} t^i \right)$$

written in holomorphic description $\mathcal{P}_{hol} \sim \gamma$, $B_{hol} = L$ takes the form

$$\mathcal{S}_\Gamma[L, \gamma, t] = \int_0^1 \Gamma^* \left(\frac{1}{2i\pi} \int_{\mathsf{c_p}} \langle L^0, d_{\mathbb{R}^N} \gamma \gamma^{-1} \rangle - H_i(L) d_{\mathbb{R}^N} t^i \right)$$

where $\mathsf{c_p}$ is a small contour around p.

Summary

$$\begin{tikzcd}[column sep=large]
\mu^{-1}(0)/\mathcal{G} \arrow[rr, leftrightarrow] \arrow[d, "\substack{\text{Action}\\ \text{3d BF}}"'] & & T^*\mathrm{Bun}_G(\mathcal{C}) \arrow[d, "\substack{\text{Unifying}\\ \text{action}}"] \\
S_\Gamma[\Sigma, \mathcal{A}] \arrow[rr, "\text{holomorphic}", "\text{description}"'] \arrow[d, "\substack{\text{univariational}\\ \text{principle}}"'] & & \mathcal{S}_\Gamma[L, \gamma, t] \arrow[d, "\substack{\text{univariational}\\ \text{principle}}"] \\
\text{Variational equations} \arrow[rr, "\text{holomorphic}", "\text{description}"'] & & \partial_{t^i} L = [M_i, L]
\end{tikzcd}$$

Examples

• In general, in genus $g \geq 2$ need to find explicit parametrisations for γ and L and then plug into our formula.

• Important well-known examples can be recovered in genus 0 and 1 but construction has to be modified: inclusion of marked points $\mathsf{p}_\alpha \in \mathcal{C}$ with coadjoint orbits attached to them. Reproduce all steps of the construction to obtain new unifying action

$$\mathcal{S}_\Gamma[L, \gamma, (\varphi_\alpha)t] = \int_0^1 \Gamma^* \left(\frac{1}{2i\pi} \int_{\mathsf{c}_\mathsf{p}} \langle L^0, d_{\mathbb{R}^N} \gamma \gamma^{-1} \rangle - H_i(L) d_{\mathbb{R}^N} t^i \right)$$
$$+ \int_0^1 \Gamma^* \sum_\alpha \langle \Lambda_\alpha, \varphi_\alpha^{-1} d_{\mathbb{R}^N} \varphi_\alpha \rangle$$

Elliptic Gaudin hierarchy: Genus 1 case, $\mathcal{C}$ torus. Fix $z(\mathsf{p}) = 0$.

- Work with Cartan-Weyl basis of $\mathfrak{g}$: $\{\mathsf{H}_\mu\}_{\mu=1}^{\mathrm{rk}\,\mathfrak{g}}$ and $\{\mathsf{E}_\varrho\}$.
- Choose transition function $\gamma = \exp\left(\frac{\mathcal{Q}}{z}\right)$, $\mathcal{Q} = q^\mu \mathsf{H}_\mu$.

Elliptic Gaudin hierarchy: Genus 1 case, $\mathcal{C}$ torus. Fix $z(\mathsf{p})=0$.

- Work with Cartan-Weyl basis of $\mathfrak{g}$: $\{\mathsf{H}_\mu\}_{\mu=1}^{\mathrm{rk}\,\mathfrak{g}}$ and $\{\mathsf{E}_\varrho\}$.
- Choose transition function $\gamma=\exp\left(\frac{\mathcal{Q}}{z}\right)$, $\mathcal{Q}=q^\mu\mathsf{H}_\mu$.
- Knowing L^0 holomorphic on U_0, L^1 meromorphic on U_1 with residues $L_\alpha=-\varphi_\alpha\Lambda_\alpha\varphi_\alpha^{-1}$ at simple poles p_α, we get

$$L^1=L_z\mathrm{d}z\,,\quad L_z=L^\mu\mathsf{H}_\mu+L^\varrho\mathsf{E}_\varrho$$

$$L^\mu=\pi^\mu+\sum_{\alpha=1}^N(L_\alpha)^\mu\zeta(z-z(\mathsf{p}_\alpha))\,,\quad \sum_{\alpha=1}^N(L_\alpha)^\mu=0$$

$$L^\varrho=\sum_{\alpha=1}^N(L_\alpha)^\varrho\frac{\sigma(\varrho(\mathcal{Q})+z-z(\mathsf{p}_\alpha))}{\sigma(\varrho(\mathcal{Q}))\sigma(z-z(\mathsf{p}_\alpha))}e^{-\varrho(\mathcal{Q})\zeta(z)}\,.$$

Elliptic Gaudin hierarchy:

- The Lagrangian one-form is

$$\mathscr{L}_{\mathrm{EG}} = p_\mu \mathrm{d}_{\mathbb{R}^n} q^\mu + \sum_{\alpha=1}^{N} \mathrm{Tr}\left(\Lambda_\alpha \varphi_\alpha^{-1} \mathrm{d}_{\mathbb{R}^n} \varphi_\alpha\right) - H_i \, \mathrm{d}_{\mathbb{R}^n} t^i \,,$$

with

$$p_\mu \coloneqq \mathrm{Tr}\left(L_z(0) \mathsf{H}_\mu\right)$$

Elliptic Gaudin hierarchy:

- The Lagrangian one-form is

$$\mathscr{L}_{\text{EG}} = p_\mu \mathrm{d}_{\mathbb{R}^n} q^\mu + \sum_{\alpha=1}^{N} \mathrm{Tr}\left(\Lambda_\alpha \varphi_\alpha^{-1} \mathrm{d}_{\mathbb{R}^n} \varphi_\alpha\right) - H_i \, \mathrm{d}_{\mathbb{R}^n} t^i \, ,$$

with

$$p_\mu \coloneqq \mathrm{Tr}\left(L_z(0)\mathsf{H}_\mu\right)$$

- Case $N = 1$ gives elliptic spin Calogero-Moser hierarchy.
- Remark: elliptic Gaudin model described before **[Nekrasov, Comm Math Phys '96]**

$\rightarrow$ Corresponds to different choice of trivialisation $A'' = a^\mu \mathsf{H}_\mu$, in which case we can show that $\gamma = 1$. Explicit comparison done and possible to derive corresponding action.

5. Perspectives and conclusions

• Lagrangian multiform theory offers *variational* description of integrability.

5. Perspectives and conclusions

• Lagrangian multiform theory offers *variational* description of integrability.

• Relation to some other integrability criterion rather well understood.

5. Perspectives and conclusions

• Lagrangian multiform theory offers *variational* description of integrability.

• Relation to some other integrability criterion rather well understood.

• Not just for integrability. A surprising observation when extending to non-commuting flows: variational description of Hamiltonian Lie group actions.

5. Perspectives and conclusions

- Lagrangian multiform theory offers *variational* description of integrability.

- Relation to some other integrability criterion rather well understood.

- Not just for integrability. A surprising observation when extending to non-commuting flows: variational description of Hamiltonian Lie group actions.

- Similar ideas of systematic construction based on classical r-matrix also work for Lagrangians two-forms for 2d ultralocal *field* theories. **[Caudrelier, Stoppato, Vicedo; Comm Math Phys '24]**

5. Perspectives and conclusions

• Lagrangian multiform theory offers *variational* description of integrability.

• Relation to some other integrability criterion rather well understood.

• Not just for integrability. A surprising observation when extending to non-commuting flows: variational description of Hamiltonian Lie group actions.

• Similar ideas of systematic construction based on classical r-matrix also work for Lagrangians two-forms for 2d ultralocal *field* theories. **[Caudrelier, Stoppato, Vicedo; Comm Math Phys '24]**

• Some open problems:

- With current (good?) understanding at classical level, how to tackle path integral quantisation of integrable hierarchy?

- At classical level, deepen understanding of the connection between Lagrangian multiforms and gauge theoretic approaches to integrability.

THANK YOU!