Lp averages of the Fourier transform in finite fields: orthogonal projections, incidence geometry, and the restriction problem

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The discrete Fourier transform is a fundamental tool in studying geometric and combinatorial problems in vector spaces over finite fields. In many cases, obtaining good uniform bounds for the Fourier transform is not possible. To address this, the notion of Lp averages of the Fourier transform - a more nuanced measure - was introduced by Fraser. We explore this idea further and discover several interesting applications.

First, we consider the problem of bounding the number of exceptional projections (projections that are smaller than typical) of a subset of a vector space over a finite field. We establish bounds that depend on Lp estimates for the Fourier transform, thereby improving various known results for sets with sufficiently good Fourier-analytic properties. The special case p = 2 recovers a recent result of Bright and Gan (following Chen), which established the finite field analogue of Peres–Schlag's bounds from the Euclidean setting.

Second, we prove several auxiliary results of independent interest, including a character sum identity for subspaces (solving a problem posed by Chen), and an analogue of Plancherel's theorem for subspaces. These auxiliary results also have applications in incidence geometry. In particular, we present a novel and direct proof of a well-known result in this area that avoids the use of spectral graph theory.

Third, we improve the finite field analogue of the Stein–Tomas restriction theorem previously established by Mockenhaupt and Tao (Duke, 2004). We generalise their result by replacing the uniform bounds on the Fourier transform with suitable Lp bounds.