A compact Riemann surface carrying regular dessins of seven different types

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Joint work with Gareth Jones

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Introduction

This talk concerns a 23yo question about highly symmetric embeddings of connected graphs on Riemann surfaces.

Here it is important to note that a Riemann surface is an orientable surface endowed with some local structure, and different local structures on the same orientable surface can give rise to different Riemann surfaces of the same genus.

The question is about how many different 'types' of highly symmetric maps and hypermaps can be carried by a given Riemann surface of genus > 0. It was shown in 2003 by Ernesto Girondo (UAM, Madrid) that the maximum number of such types is at most 7, and he asked if 7 is possible. I'll show that the answer is "Yes", so 7 is the maximum possible.

Formally, a compact Riemann surface S is a quotient space $\mathcal{U}/\Lambda$, where $\mathcal{U}$ is the upper-half complex plane, and Λ is a co-compact Fuchsian group, which is a finitely-generated torsion-free discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, isomorphic to the fundamental group of S.

The full automorphism group $\mathrm{Aut}(S)$ of S, which includes orientation-preserving and (possibly) orientation-reversing automorphisms, is isomorphic to the quotient Γ/Λ, where Γ is the normaliser of Λ in $\mathrm{PGL}(2,\mathbb{R})$.

The conformal (orientation-preserving) automorphism group $\mathrm{Aut}^+(S)$ of S is isomorphic to Γ^+/Λ, where Γ^+ is the corresponding subgroup of index at most 2 in Γ, and equal to the normaliser of Λ in $\mathrm{PSL}(2,\mathbb{R})$. Also the well-known Riemann-Hurwitz formula relates the $|\mathrm{Aut}^+(S)|$ to the genus g of S via the area of a fundamental domain for Γ^+.

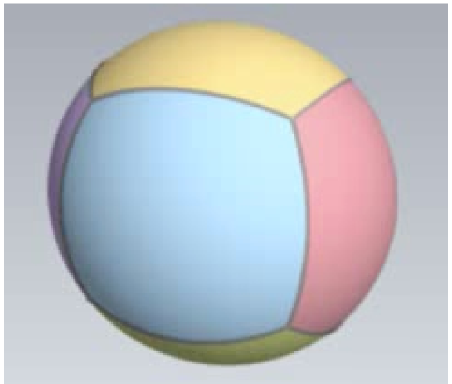
Orientably-regular maps and hypermaps

Let S be a closed orientable surface (such as the sphere, or the torus, or the double torus, and so on).

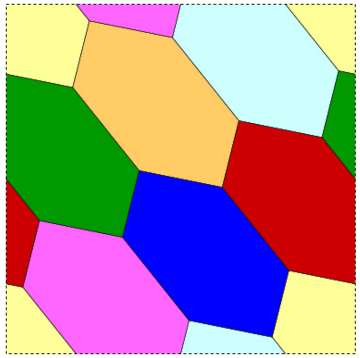
A map on S is an embedding of a connected graph on S, breaking S into faces that are simply-connected (and hence homeomorphic to open disks). An automorphism of a map $\mathcal{M}$ is a bijection from $\mathcal{M}$ to itself that preserves its vertices, its edges and its faces, and incidences between them.

Such a map $\mathcal{M}$ is orientably-regular if its group $\mathrm{Aut}^+(\mathcal{M})$ of orientation-preserving automorphisms has acts transitively on arcs (= ordered pairs of adjacent vertices). In that case every vertex has the same valency, say k, and every face has the same co-valency (= number of edges), say m, and then we call the pair $\{m, k\}$ the type of $\mathcal{M}$.

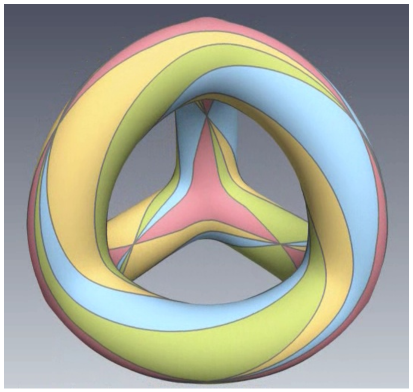
Examples:



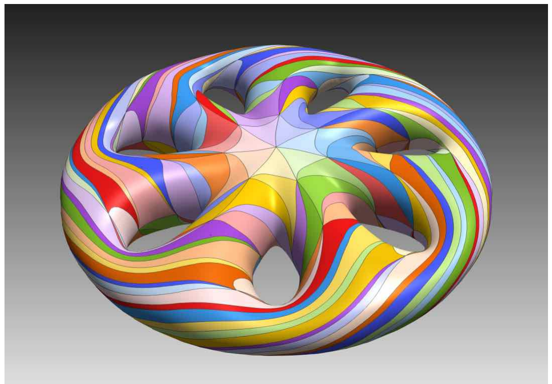
Genus 0, type $\{4, 3\}$



Genus 1, type $\{6, 3\}$



Genus 3, type $\{3, 8\}$



Genus 7, type $\{7, 3\}$

Maps can be generalised to hypermaps, which may be viewed as embeddings of (connected) hypergraphs on surfaces, but with no requirement that a hyperedge is incident with at most two hypervertices and at most two hyperfaces.

A dessin (or 'dessin d'enfant', as termed by Grothendieck) is simply a finite bipartite map, namely an embedding of a connected finite bipartite graph into an orientable surface. This becomes a hypermap if we consider the vertices in one part of the bipartite graph as the hypervertices, the vertices in the other part as the hyperedges, and the faces of the bipartite map as the hyperfaces.

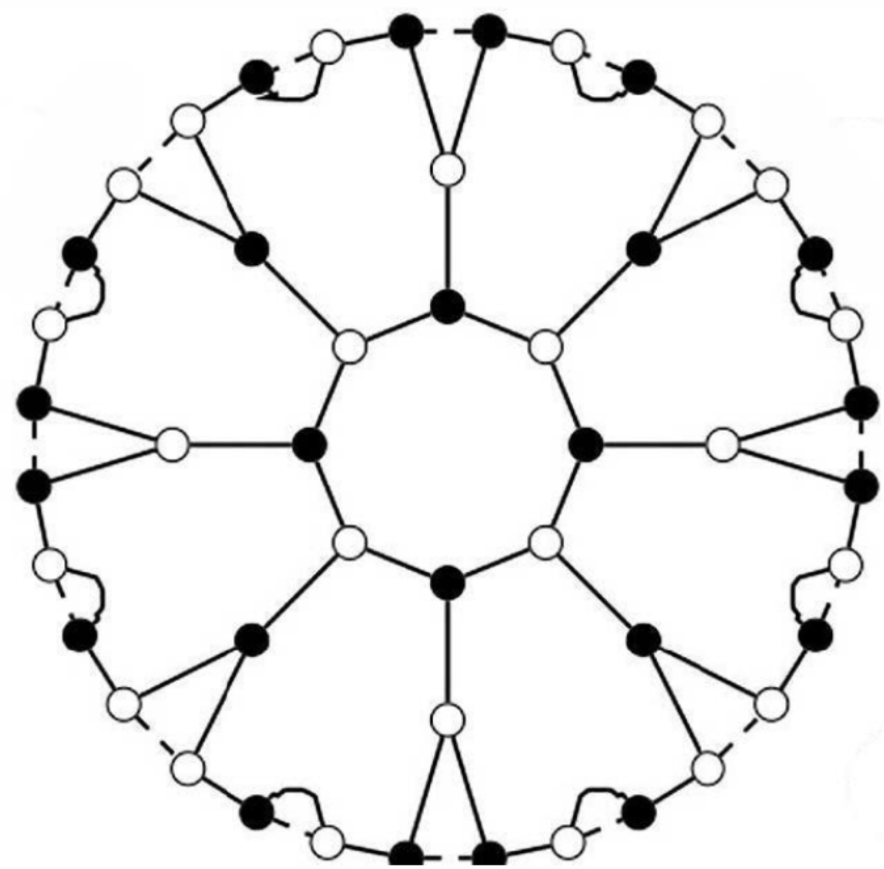
An automorphism of a dessin $\mathcal{D}$ is an orientation-preserving bijection from $\mathcal{D} \to \mathcal{D}$, preserving the sets of hypervertices, hyperedges and hyperfaces, and incidences between them.

A dessin $\mathcal{D}$ is said to be regular if the group Aut($\mathcal{D}$) of all automorphisms of $\mathcal{D}$ acts regularly (sharply-transitively) on the set of edges of the corresponding bipartite graph. In that case every hypervertex has the same valency k, every hyperedge has the same valency l, and every hyperface has the same size $2m$, and we say that $\mathcal{D}$ has type (k, l, m).

Also Aut($\mathcal{D}$) is a 'smooth' quotient of the ordinary triangle group $\Delta^+(k, l, m) = \langle\, x, y, z \mid x^k = y^l = z^m = zyz = 1 \,\rangle$ and the genus g and Euler characteristic χ of $\mathcal{D}$ are given by $2 - 2g = \chi = |\mathrm{Aut}(\mathcal{D})|(1/k + 1/l + 1/m - 1)$. Conversely, every finite smooth quotient of $\Delta^+(k, l, m)$ gives such a $\mathcal{D}$.

If $l = 2$ then $\mathcal{D}$ corresponds to a map $\mathcal{M}$, and $\mathcal{D}$ is orientably-regular with $\mathrm{Aut}(\mathcal{D}) = \mathrm{Aut}^+(\mathcal{M})$ in the terminology of maps. So a regular dessin is an orientably-regular map or hypermap.

Example: Walsh map of a regular dessin of type $(3, 3, 4)$



Reflexibility and chirality

Note that a regular dessin $\mathcal{D}$ could be reflexible (admitting an orientation-reversing bijection $\mathcal{D} \to \mathcal{D}$, preserving the sets of hypervertices, hyperedges and hyperfaces, and incidences between them). For example, all regular dessins on the sphere (of genus 0) have type $(2,2,n)$ for some $n \geq 2$, or $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$, and all of them are reflexible.

If there exists no such orientation-reversing bijection $\mathcal{D} \to \mathcal{D}$, then $\mathcal{D}$ is chiral. For example, there are infinitely many regular dessins on the torus (of genus 1), and all have type $(2,3,6)$, $(2,4,4)$ or $(3,3,3)$, and some are reflexible, but almost all are chiral.

Girondo's question

A compact Riemann surface is called quasiplatonic if it carries a regular dessin (and multiply quasiplatonic if it carries more than one regular dessin).

In 2003 Ernesto Girondo showed that the largest number of distinct types of regular dessins that can be carried by a given compact Riemann surface of genus $g > 0$ is **at most seven**, and asked if seven is possible.

We have answered this, and more, as follows ...

Theorem [MC & Gareth Jones, 2026]

(a) There exists a quasiplatonic surface S of genus 721 carrying regular dessins of seven distinct types, the maximum possible for such a surface of given genus $g > 0$.

(b) The types in (a) are $(2,3,8)$, $(2,4,8)$, $(2,8,8)$, $(3,3,4)$, $(3,8,8)$, $(4,4,4)$ and $(4,8,8)$.

(c) S is the unique such surface of minimum possible $g > 0$.

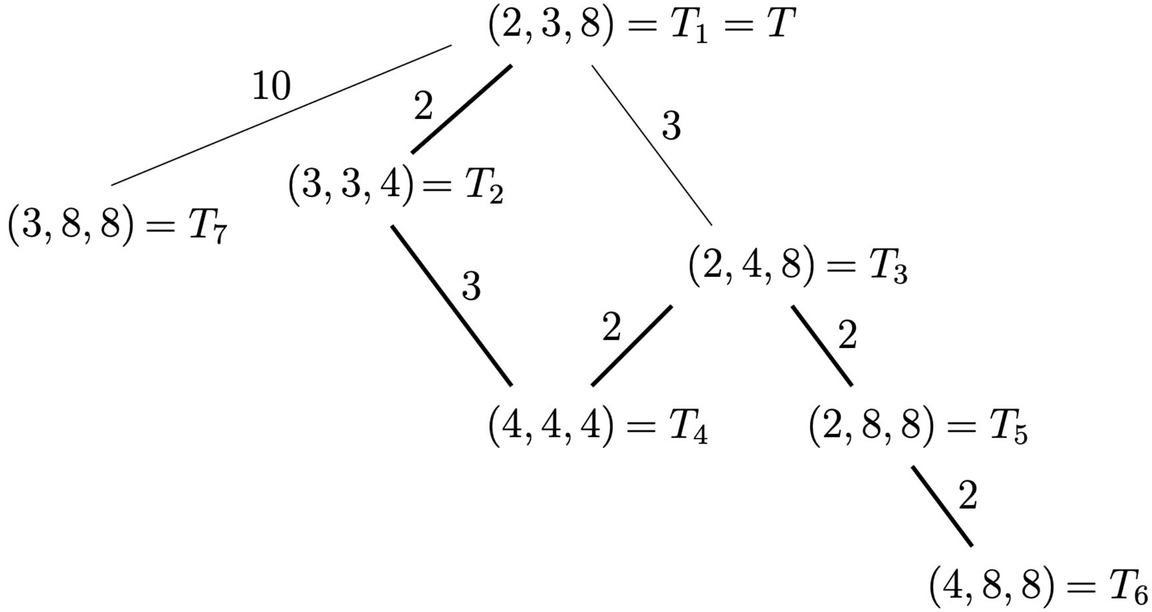
(d) The regular dessin of type $(3,8,8)$ is chiral, while all of the other six are reflexible.

(e) All other such surfaces are covers of S, carrying regular dessins of the same seven types.

(f) There exist such surfaces of infinitely many genera.

Key steps in the proof

The existence of S is based on showing that PSL$(2,\mathbb{R})$ contains subgroups isomorphic to each of the ordinary triangle groups $\Delta^+(2,3,8)$, $\Delta^+(2,4,8)$, $\Delta^+(2,8,8)$, $\Delta^+(3,3,4)$, $\Delta^+(3,8,8)$, $\Delta^+(4,4,4)$ and $\Delta^+(4,8,8)$, as below:



These subgroups have a common intersection of index 120 in $\Delta^+(2,3,8)$, the core of which is a torsion-free normal subgroup of index 34560 in $\Delta^+(2,3,8)$. The latter normal subgroup K is the fundamental group of the surface S whose genus g and Euler characteristic χ are given by

$$2 - 2g = \chi = 34560(1/2 + 1/3 + 1/8 - 1) = -1440$$

so the genus of S is 721. This proves (a) and (b).

The uniqueness of S (part (c)) follows from some work by Kisao Takeuchi (1977) on commensurabilty classes of arithmetic triangle groups, and from David Singerman's theory (from 1972) of finitely maximal Fuchsian groups.

Reflexibility/chirality of the seven regular dessins carried by S can be determined with the help of MAGMA, giving (d).

Finally, part (e) about covers follows immediately from (c), and then finally, the 'Macbeath trick' (from 1969) for constructing covers with abelian covering group shows that for every positive integer n there exists a cover of S of genus $1 + 720n^{1442}$ that carries seven regular dessins of the same types as the seven dessins on S.

Additional facts proved

- There are also infinitely many quasiplatonic surfaces that carry regular dessins of the same 7 types that are all chiral.
- The largest number of types of reflexible regular dessins that can be carried by a given compact Riemann surface of genus $g > 0$ is exactly six, occurring on a unique surface of genus 3, and on infinitely many covers of this surface.
- This bound of six is also attained using $\Delta^{+}(2, 3, 12n)$ and its triangular subgroups of types $(3, 3, 6n)$, $(2, 6n, 12n)$, $(3, 4n, 12n)$, $(6n, 6n, 6n)$ and $(3n, 12n, 12n)$, for all n.
- The largest number of types of orientably-regular maps that can be carried by a given compact Riemann surface of genus $g > 0$ is exactly three, occurring on a unique surface of genus 2, and on infinitely many covers of this surface.

Thank you for listening!

Title: Title: A compact Riemann surface carrying regular dessins of seven different types

Speaker: Prof. Marston Conder, University of Auckland (New Zealand) and University of Primorska (Slovenia)

Abstract: In this short talk I'll describe the answer to a 2003 question by Ernesto Girondo, by showing that there is a 'quasiplatonic' Riemann surface S of genus 721 that carries regular dessins (orientably-regular maps or hypermaps) of seven different types, the maximum number that was thought could be possible for genus greater than 0. In fact S is the unique such surface of minimum possible genus greater than 0, and six of the seven regular dessins on S are reflexible, while the seventh is chiral. Also there are infinitely many such surfaces, all occurring as covers of S. Some of this is joint work with Gareth Jones (Southampton).