Twist polynomial interpolation for binary delta-matroids

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This is joint work with Zhao Zhao.

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Partial-dual polynomial

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The coefficient of z^k counts partial duals of Euler genus k.

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Equivalently, there are no gaps in the even part or in the odd part.

Example of even and odd interpolation

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Even degrees

$$\{4, 6\}$$

consecutive among even degrees

even-interpolating

Odd degrees

$$\{1, 3, 5\}$$

consecutive among odd degrees

odd-interpolating

Known result for orientable ribbon graphs

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What happens for non-orientable ribbon graphs?

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Signed rotation	**Partial-dual polynomial**	**Euler genera**
$(-1,-2,3,1,4,2,$ $5,4,3,6,5,6)$	$2z+14z^3+12z^4+28z^5+8z^6$	$\{1,\cancel{2},3,4,5,6\}$
$(-1,2,3,2,4,3,5,$ $6,7,1,6,7,4,5)$	$8z^2+48z^4+16z^5+40z^6+16z^7$	$\{2,\cancel{3},4,5,6,7\}$
$(-1,2,3,2,4,5,6,7,$ $8,1,7,8,5,6,3,4)$	$16z^2+80z^4+112z^6+32z^7+16z^8$	$\{2,\cancel{3},4,\cancel{5},6,7,8\}$

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Once two consecutive Euler genera appear, no further gaps occur up to the maximum.

Gross–Mansour–Tucker problem

Problem: even–odd interpolation

For a non-orientable ribbon graph G, under what interesting conditions is

$$^{\partial}\varepsilon_G(z)$$

even-interpolating, odd-interpolating, or both?

Gross–Mansour–Tucker, 2020

Twist and width for delta-matroids

Let $D = (E, \mathcal{F})$ be a delta-matroid.

Twist

For $A \subseteq E$, the twist of D with respect to A is

$$D * A = (E, \{ A \Delta X \mid X \in \mathcal{F} \}).$$

Width

Let $r_{\max}(D)$ and $r_{\min}(D)$ be the maximum and minimum sizes of feasible sets.

$$\omega(D) = r_{\max}(D) - r_{\min}(D).$$

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Twisting gives a family of delta-matroids, and width records their sizes.

From ribbon graphs to delta-matroids

Let $G = (V, E)$ be a connected ribbon graph. Define $D(G) = (E, \mathcal{F}(G))$ where

$$\mathcal{F}(G) = \{\, A \subseteq E \mid (V, A) \text{ is a spanning quasi-tree of } G \,\}.$$

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Chun–Moffatt–Noble–Rueckriemen, 2019: Ribbon graphs can be studied via delta-matroids.

Ribbon graphs		**Delta-matroids**
partial duality	$\longleftrightarrow$	twist
Euler genus	$\longleftrightarrow$	width

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$$^{\partial}\omega_{D(G)}(z) = {}^{\partial}\varepsilon_G(z)$$

Main theorem for binary delta-matroids

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Theorem Zhao–Yan, 2026+

Let $D = (E, \mathcal{F})$ be a binary delta-matroid. If

$$W(D) := \{\, \omega(D * A) \mid A \subseteq E \,\}$$

contains two consecutive integers k and $k+1$, then

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Consequently, ${}^{\partial}\omega_D(z)$ is either an even polynomial, an odd polynomial, or both even-interpolating and odd-interpolating.

The binary assumption is necessary

Counterexample

$$E = \{1, 2, \ldots, 8\}, \quad D = (E, \mathcal{F}), \quad \mathcal{F} = \{F \subseteq E \mid |F| \in \{0, 1, 3, 4\}\}.$$

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$$^{\partial}\omega_D(z) = 2z^4 + 72z^5 + 112z^7 + 70z^8$$

$$\text{supp}_{\text{even}} = \{4, 8\}, \qquad \boxed{6 \text{ is missing}}.$$

$$\Longrightarrow \qquad {}^{\partial}\omega_D(z) \text{ is } \textbf{not even-interpolating}.$$

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Thus the **binary assumption** **cannot be omitted**.

Consequence for ribbon graphs

The maximum partial-dual Euler-genus of a ribbon graph G is

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Thank you for your attention!