The Charlton Li formula for the twist polynomial of wedge sum of binary Δ-matroids

Sergei Chmutov

Ohio State University, Mansfield

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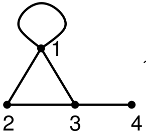
Binary Δ-matroids

(Looped) simple graph G ⇝ Adjacengy matrix M over $\mathbb{F}_2$

$G =$



⇝ $M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(Normal) Δ-matroid of a graph, $D_G = (E; \mathcal{F})$: $E :=$ set of vertices of G; $F \in \mathcal{F}$ iff the principal minor of M indexed by F is non-degenerate.

Example. $E := \{1, 2, 3, 4\}$,

$$\mathcal{F} := \Big\{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\Big\}.$$

Definition. A *binary* Δ-matroid is a twist of D_G relative to a subset $A \subset E$, $D_G * A$.

Twists of Δ-matroids. Let $D=(E;\mathcal{F})$ be a Δ-matroids and $A\subseteq E$.

$$D*A := (E;\{F\Delta A | F\in\mathcal{F}\}).$$

$D_{min} := (E;\mathcal{F}_{min})$, where $\mathcal{F}_{min} := \{F\in\mathcal{F} | F$ is of minimal possible cardinality$\}$.

$D_{max} := (E;\mathcal{F}_{max})$, where $\mathcal{F}_{max} := \{F\in\mathcal{F} | F$ is of maximal possible cardinality$\}$.

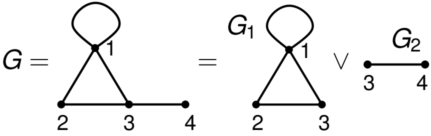
D_{min} and D_{max} are genuine matroids. $\boxed{w(D) := r_{D_{\max}}(E) - r_{D_{\min}}(E)}$

The twist polynomial of a Δ-matroid $D=(E,\mathcal{F})$ is the generating function for the width of all twists of D, ${}^{\partial}w_D(z) := \sum\limits_{A\subseteq E} z^{w(D*A)}$.

Example.

$$\begin{aligned} {}^{\partial}w_{D_G}(z) &= \overset{A=\emptyset,\{1,2,3,4\}}{2z^4} + \underset{A=\{1\},\{2,3,4\}}{2z^3} + \overset{A=\{2\},\{1,3,4\}}{2z^3} + \underset{A=\{3\},\{1,2,4\}}{2z^2} + \overset{A=\{4\},\{1,2,3\}}{2x^3} + \underset{A=\{1,2\},\{3,4\}}{2z^4} + \overset{A=\{1,3\},\{2,4\}}{2z^2} + \underset{A=\{1,4\},\{2,3\}}{2z^3} \\ &= 4z^4 + 8z^3 + 4z^2\ . \end{aligned}$$

Definition. Let G_1 and G_2 be looped simple graphs with respective vertices v_1 and v_2 that are either both unlooped or both looped. Define the *wedge sum* $G_1 \vee G_2$ to be the disjoint union of G_1 and G_2 with the vertices v_1 and v_2 identified.



Q.Yan, X.Jin, *Partial-dual polynomials and signed intersection graphs*, **Forum of Mathematics**, 10:e69 (2022):
Theorem 5.2. Let G be a looped simple graph with a pendant edge (v_1, v_2) where v_1 is unloopped and has degree 1; $G = G_1 \vee G_2$, where G_2 is a single edge as in the example above. Then ${}^{\partial}w_{D_G}(z) = {}^{\partial}w_{D_{G_1}}(z) + (2z^2)\,{}^{\partial}w_{D_{G_1 - v_2}}(z)$.

Example. ${}^{\partial}w_{D_{G_1}}(z) = 4z^3 + 4z^2$, ${}^{\partial}w_{D_{G_1 - v_2}}(z) = 2z^2 + 2z$. So ${}^{\partial}w_{D_G}(z) = 4z^3 + 4z^2 + (2z^2)(2z^2 + 2z) = 4z^4 + 8z^3 + 4z^2$.

C.Li, *A recursion for the twist polynomial of a one-point join of normal binary delta-matroids*, **arXiv:2511.09504v1 [math.CO]**.
Assume that G_1 and G_2 are looped simple graphs with respective vertices v_1 and v_2 that are either both unlooped or both looped and either $G_1 - v_1$ or $G_2 - v_2$ is unlooped. Then:
Theorem 5.

$$(2z^2-2)^{\partial}w_{D_{G_1\vee G_2}}(z) = [^{\partial}w_{D_{G_1}}, ^{\partial}w_{D_{G_1-v_1}}]\begin{bmatrix} -1 & 2z^2 \\ 2z^2 & -4z^2 \end{bmatrix}\begin{bmatrix} ^{\partial}w_{D_{G_2}} \\ ^{\partial}w_{D_{G_2-v_2}} \end{bmatrix}.$$

Example. If G_2 is a single edge as in the example above, then $^{\partial}w_{D_{G_2}} = 2z^2+2$ and $^{\partial}w_{D_{G_2-v_2}} = 2$. The formula in the Theorem became the Yan – Jin formula for $^{\partial}w_{D_{G_1\vee G_2}}$.

Definition. For a simple looped graph G with a vertex v, the *loop complementation* of G at v is the graph $G+v$ obtained from G by adding (resp. removing) a loop at v if v was originally unlooped (resp. looped) in G. Let G_1 and G_2 be looped simple graphs with respective vertices v_1 and v_2 that are either both unlooped or both looped and either G_1-v_1 or G_2-v_2 is unlooped. Consider a row-vector $\boldsymbol{v}_1 := [{}^{\partial}w_{D_{G_1}}, {}^{\partial}w_{D_{G_1+v_1}}, {}^{\partial}w_{D_{G_1-v_1}}]$ and the column vector $\boldsymbol{v}_2 := \begin{bmatrix} {}^{\partial}w_{D_{G_2}} \\ {}^{\partial}w_{D_{G_2+v_2}} \\ {}^{\partial}w_{D_{G_2-v_2}} \end{bmatrix}$.Then:

Theorem 4.

$$(4z^2-2z-2)\,{}^{\partial}w_{D_{G_1\vee G_2}}(z) = \boldsymbol{v}_1 \begin{bmatrix} -(z+1) & z & 2z^2 \\ z & -(z+1) & 2z^2 \\ 2z^2 & 2z^2 & -4z^2(z+1) \end{bmatrix} \boldsymbol{v}_2\ .$$

Corollary 20. Loop complementation formula. *If G is an unlooped simple graph and v is any vertex of G, then* ${}^{\partial}w_{D_{G+v}} = \dfrac{z\,{}^{\partial}w_{D_G} + (2z^2)\,{}^{\partial}w_{D_{G-v}}}{z+1}$.

Proposition 28. Pendant edge formula for general Δ-matroids. *Let $D = (E, \mathcal{F})$ be a normal Δ-matroid with distinct $e_1, e_2 \in E$ such that $\{e_1, e_2\} \in \mathcal{F}$ and $D \setminus e_2 = (\{e_1\}; \{\emptyset\}) \oplus D \setminus \{e_1, e_2\}$. Then*

$$ {}^{\partial}w_D(z) = {}^{\partial}w_{D \setminus e_1}(z) + (2z^2)\,{}^{\partial}w_{D \setminus \{e_1, e_2\}}(z) \; . $$

Description of the feasible sets of the wedge product in terms of the feasible sets of its components.

- Is it possible to define a wedge product of general Δ-matroids; or of some strictly larger class of Δ-matroids, vf-safe for instance, in such a way that the Theorem 5 would hold for it?
- What is the meaning/significance of the bilinear forms of Theorems 4 and 5?
- The twist polynomial of a Δ-matroid can be read of the Bollobás–Riordan polynomial of the Δ-matroid. Are there any analogs of Theorems 4 and 5 for the Bollobás–Riordan polynomial of Δ-matroids instead of the twist polynomial?

THANK YOU!!!