A survey of digraph structure theory

Paul Seymour (Princeton)

Two connecting paths problem: Given vertices s_1, t_1, s_2, t_2 of a graph, decide whether there are vertex-disjoint paths P_1, P_2 where P_i joins s_i, t_i for $i = 1, 2$.

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Theorem (Fortune, Hopcroft and Wyllie (1980))

This is NP-complete.

Theorem (Bartell, S. 2025)

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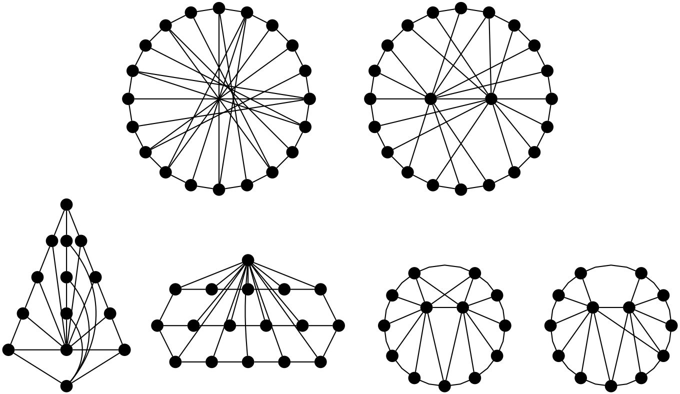


Say G is *just-nonplanar* if G is 3-connected, non-planar, and F is the edge-set of a forest, where F is the set of edges e such that $G \setminus e$ is nonplanar.

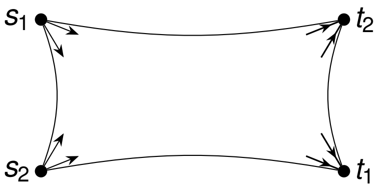
Theorem

If G is a digraph that is minimal (under subdigraph containment) strong and nonplanar, and not the directed subdivision of a smaller digraph, then it is an orientation of a just-nonplanar graph.

Theorem (Thomassen (1985))

Let G be an acyclic digraph, and let $S = \{s_1, s_2\}$ be its set of sources, and $T = \{t_1, t_2\}$ its set of sinks. Suppose that $S \cap T = \emptyset$, and every vertex not in $S \cup T$ has in-degree and out-degree both at least two. Then exactly one of the following holds:

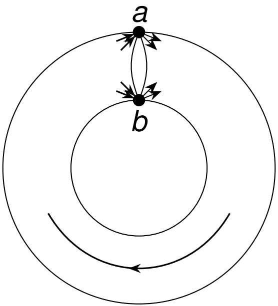
- *there are vertex-disjoint directed paths P_1, P_2 of G, where P_i is from s_i to t_i for $i = 1, 2$;*
- *G can be drawn in a closed disc with s_1, s_2, t_1, t_2 drawn in the boundary in order.*



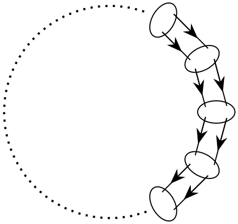
Say a digraph is circular if it can be drawn in the plane such that every directed cycle winds clockwise around the origin.
Thomassen's theorem says:

Theorem

Let G be a digraph such that every vertex has in-degree and out-degree at least two. If $a, b \in V(G)$, and every directed cycle contains exactly one of a, b, then G is circular.



If all directed cycles have length divisible by ℓ, G looks like:



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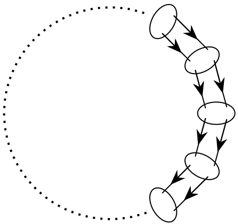
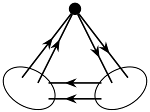


Theorem (S. (2025))

Digraphs in which all directed cycles have length three are built by pasting together digraphs as below.



Checking if a digraph has an odd directed cycle is easy. How to check if it has an even directed cycle?

G is odd-weightable if there is some integer edge-weighting such that every directed cycle has odd total weight.

Theorem

The following problems are polynomially equivalent:

- *Does G have an even directed cycle?*
- *Given an integer edge-weighting of G, does some directed cycle have even total weight?*
- *Is G odd-weightable?*

G is odd-weightable if there is some integer edge-weighting such that every directed cycle has odd total weight.

A double-cycle of length k is the union of k directed cycles $C_1, \ldots, C_k$, where $k \geq 3$ and $C_i \cap C_{i+1}$ is a path for $1 \leq i \leq k$, and C_i, C_j are disjoint if $j \neq i \pm 1$.

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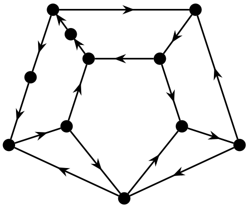

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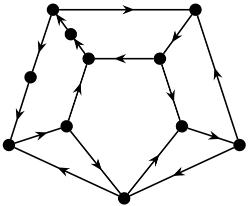
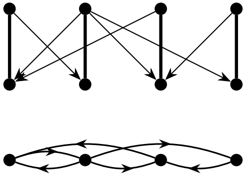


Theorem (S., Thomassen (1987))

A digraph is odd-weightable iff it has no odd double-cycle as a subdigraph.

Any digraph can be obtained by:
Take a graph with a bipartition (A, B), direct all edges from A to B, choose a perfect matching M and contract M.



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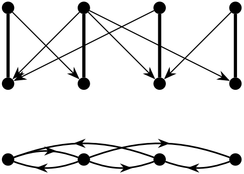

Amazing fact: If G is a bipartite graph, and contracting some perfect matching makes an odd-weightable digraph, then contracting any perfect matching also makes an odd-weightable digraph.

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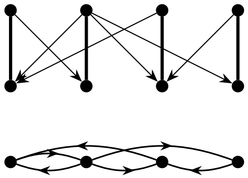

Amazing fact: If G is a bipartite graph, and contracting some perfect matching makes an odd-weightable digraph, then contracting any perfect matching also makes an odd-weightable digraph.

Theorem (Robertson, S., Thomas (1997))

The bipartite graphs with perfect matchings whose contraction makes an odd-weightable digraph are those made by sum operations from planar bipartite graphs and the Heawood graph.

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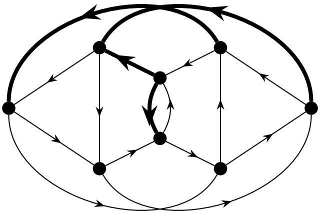
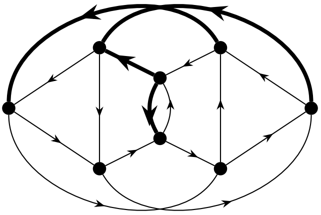
Theorem (Berger, Carter, S. (2025))

The following are equivalent:

- *there is a real-valued edge-weighting such that all directed cycles have total weight one;*
- *there is a 0/1-valued edge-weighting such that all directed cycles have total weight one;*
- *for every triple of vertices a, b, c, all directed cycles containing all of a, b, c pass through them in the same cyclic order;*
- *G has no subdigraph that is a double-cycle.*





Theorem (Berger, Carter, S. (2025))

The strong weightable digraphs are the digraphs built from strong planar weightable digraphs by sum operations.

Which strong planar digraphs are weightable?

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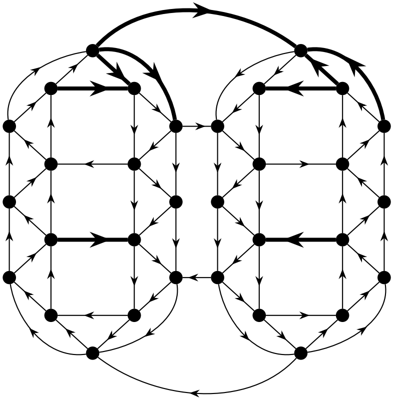
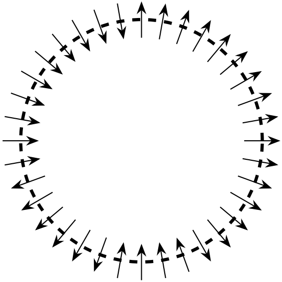


Bond: minimal edge-cutset. If G is drawn in the plane, the edges in a bond are in a circular order. So if G is a planar digraph, bonds have "change numbers".



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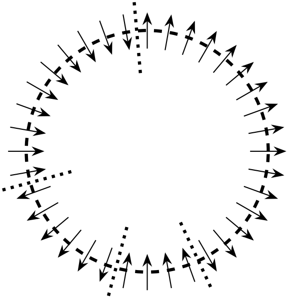

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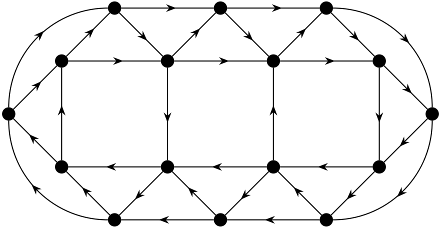

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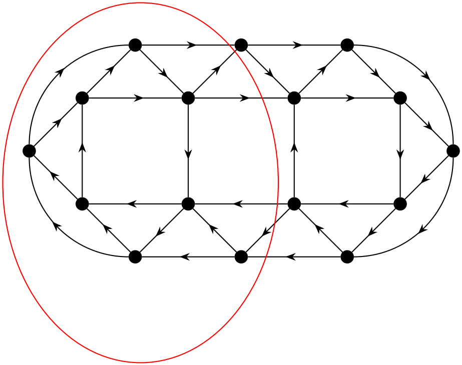

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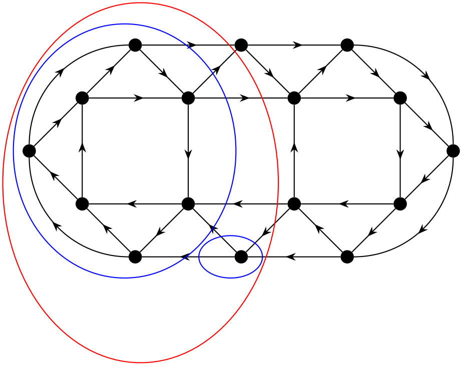

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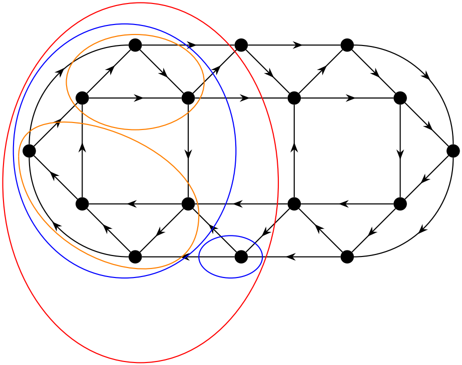

Bond carving: bijection τ from $V(G)$ to the leaves of a subcubic tree T, such that for $e \in E(T)$, the set of edges of G joining vertices mapped to different components of $T \setminus e$ is a bond. (Only 2-connected graphs admit bond carvings.)

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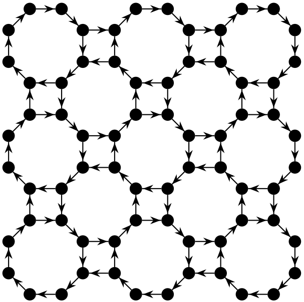
A weakly 2-connected planar digraph G has diwidth $\leq k$ if G admits a bond carving where all corresponding bonds have change number at most k.

The 6×6 diwall:



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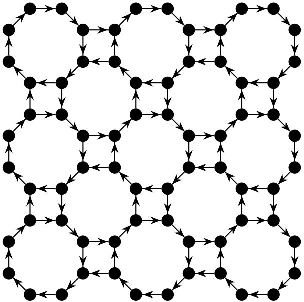
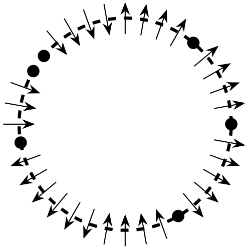


Theorem (Chudnovsky, S. (2025))

A weakly 2-connected plane digraph with bounded vertex-change-number has large diwidth iff some subdigraph is a directed subdivision of a large diwall.

Dart-width: Version of diwidth where the cutsets are allowed to use a bounded number of vertices as well as a bounded number of intervals of edges.



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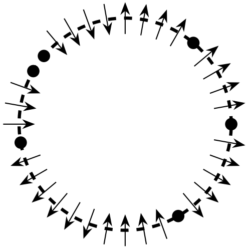

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A loopless, weakly 2-edge-connected plane digraph has large dart-width iff some subdigraph is a directed subdivision of a large diwall.

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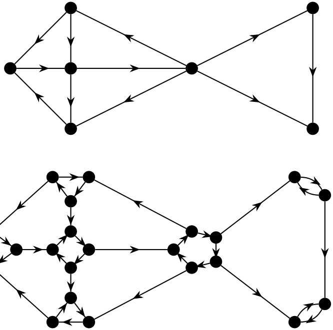

Digraph H is a semi-strong minor of digraph G if a directed subdivision of H can be obtained from a subdigraph of G by contracting strongly-connected subgraphs to vertices.

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Theorem (Chudnovsky, S. (2025))

For every planar digraph H there exists k such that every loopless, weakly 2-edge-connected loopless plane digraph that does not contain H as a semi-strong minor has dart-width at most k.

Theorem (Kawarabayashi, Kreutzer (2014))

For every circular digraph H, every digraph with sufficiently large "directed tree-width" contains H as a "butterfly minor".

Butterfly minor: Delete arbitrarily. Contracting an edge uv is only allowed if either u has outdegree one or v has indegree one.

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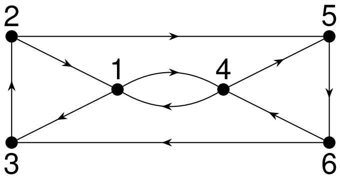

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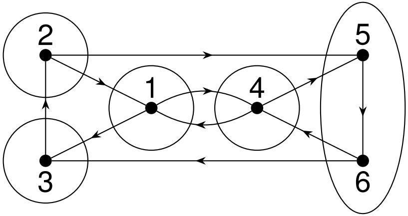

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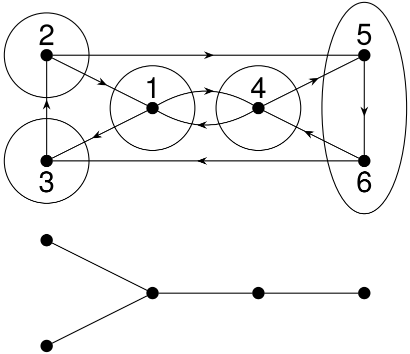

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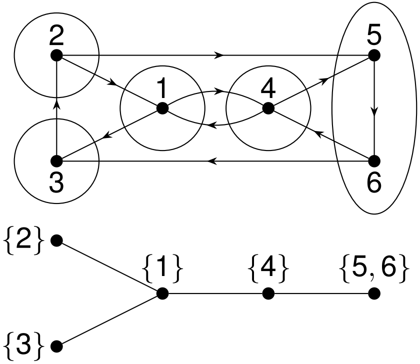

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