Matroid Products and Combinatorial Rigidity

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Joint work with Shin-ichi Taigawa

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Symmetric Power of a matroid

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Multilinearity For all $x \in V \setminus L_M$, the map $x \cdot v \mapsto v$ is an isomorphism from $N|_{x \cdot V}$ to M and, for all $x \in L_M$, $x \cdot V$ is a set of loops of N.

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Anderson (2024) considered matroids satisfying Mult+PRP, proved that all such matroids satisfy FP, and defined N to be a **second symmetric power of** M if it satisfies Mult+PRP (+FP). He also observed that a second symmetric power of M may not be unique, and may not even exist when M is not realisable.

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We will verify Mason's conjecture. In addition we show that the family of matroids considered by Lovász is a strictly larger family than Anderson's second symmetric powers.

Theorem If N satisfies Mult+FP with respect to M, then N satisfies PRP. In addition, there exists a matroid N which satisfies PRP+FP with respect to M but does not safisfy Mult.

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Theorem If N satisfies Mult+FP with respect to M, then N satisfies PRP. In addition, there exists a matroid N which satisfies PRP+FP with respect to M but does not safisfy Mult.

Our proof uses techniques from combinatorial rigidity theory to verify the dual statement to this theorem.

Dual Properties

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Cyclic Property For all cyclic sets X of M, $\mathrm{Sym}_2(X)$ is a cyclic set of N.

Relation to rigidity

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(RRP) $\operatorname{rank} N = \binom{n+1}{2} - \binom{n-\operatorname{rank} M+1}{2}$, when $|V| = n$.

(CP) For all cyclic sets X of M, $\mathrm{Sym}_2(X)$ is a cyclic set of N.

The above properties recall properties from combinatorial rigidity theory. More precisely, when M is the uniform matroid U_n^d and N is a matroid on the edge set of the looped complete graph K_n^o:

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CP tells us that the edge set of every copy of K_{d+1}^o is a cyclic set in N.

Extension Property and Main Result

We need to consider one further property for matroids M on V and N on $\mathrm{Sym}_2(V)$. For $x \in V$ and $I \subseteq \mathrm{Sym}_2(V)$ let

$$\mathrm{lk}(x, I) = \{y \in V : xy \in I\}.$$

ExtP For every N-independent set $I \subseteq \mathrm{Sym}_2(V)$ and all $v, x \in V$ such that $v \notin \mathrm{cl}_M(\mathrm{lk}(x, I))$, $I + xv$ is N-independent.

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We use this new operation to verify the dual form of Mason's Conjecture.

Theorem The following statements for matroids M on V and N on $\mathrm{Sym}_2(V)$ are equivalent.

- N satisfies DMult and CycP wrt M.
- N satisfies RRP and ExtP wrt M.

Mason's Conjecture for Higher Powers

Given a matroid M on V, we can extend the properties Mult, PRP and FlatP to a matroid N on $\mathrm{Sym}_k(V)$. Mason conjectured that Mult and FlatP imply PRP for all k. We can show that this conjecture is false when $k \geq 3$, that it is true when $k = 3$ and M is a uniform matroid other than the free matroid, and that it is false for non-free uniform matroids when $k = 4$.

Mason's Maximality Question

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It is difficult to imagine what a free-est second symmetric power may be, even in the case when M is realisable. For example, when $M = U_n^d$, we can construct a second symmetric power of M by first realising M as a set of n generic vectors in $\mathbb{R}^d$ and then using the symmetric tensor product of pairs of vectors as a realisation of a second symmetric power of M. This gives the free-est second symmetric power of U_n^d when $d = 1, 2$ but we can use a matroid from rigidity theory to show it is not the free-est when $d \geq 7$.