

A Short Survey of Lattice-Isomorphism-Based Cryptography

Wessel van Woerden (PQShield).



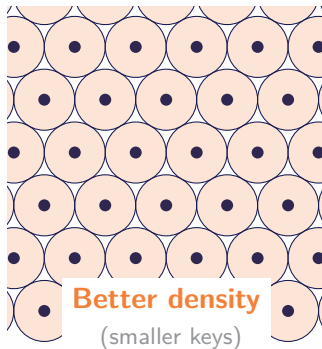
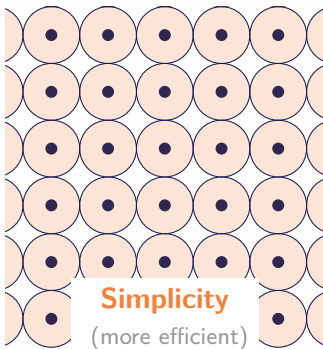
Motivation

LWE, SIS, NTRU lattices: **versatile**, but **poor geometry & algorithms**.

Motivation

LWE, SIS, NTRU lattices: **versatile**, but **poor geometry & algorithms**.

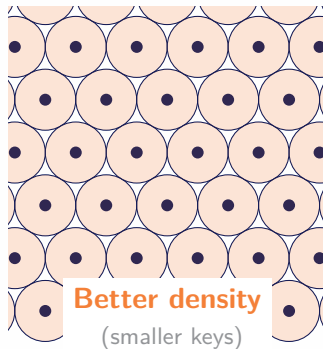
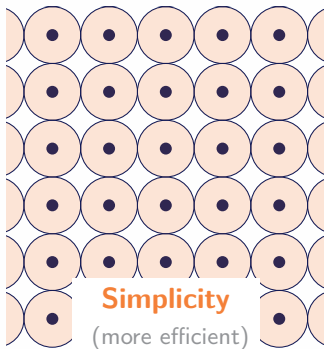
Many remarkable lattices exist with great geometric or algorithmic properties.



Motivation

LWE, SIS, NTRU lattices: **versatile**, but **poor geometry & algorithms**.

Many remarkable lattices exist with great geometric or algorithmic properties.

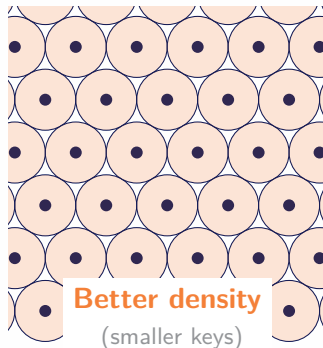
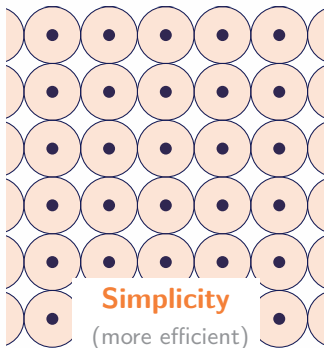


Can we use these in cryptography?

Motivation

LWE, SIS, NTRU lattices: **versatile**, but **poor geometry & algorithms**.

Many remarkable lattices exist with great geometric or algorithmic properties.

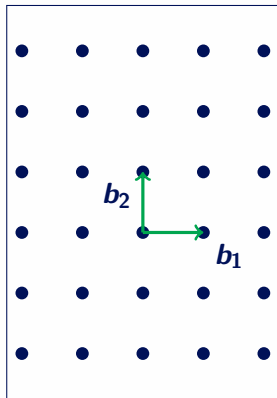


Can we use these in cryptography?

Lattice Isomorphism Problem: yes, we can!

Lattices and decoding

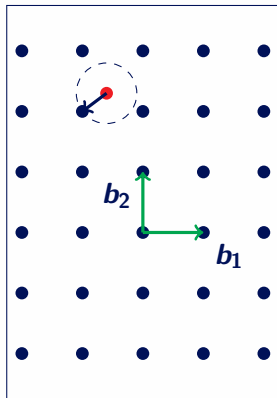
Lattice $\mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$



integer lattice $\mathbb{Z}^n$

Lattices and decoding

$$\text{Lattice } \mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$

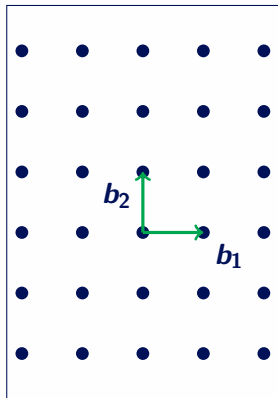


integer lattice $\mathbb{Z}^n$

decoding easy

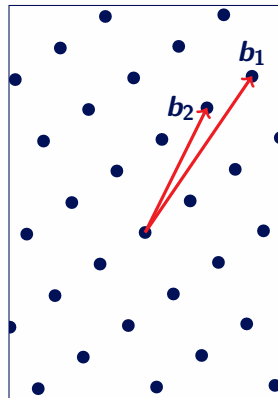
Lattices and decoding

$$\text{Lattice } \mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$



integer lattice $\mathbb{Z}^n$

decoding easy

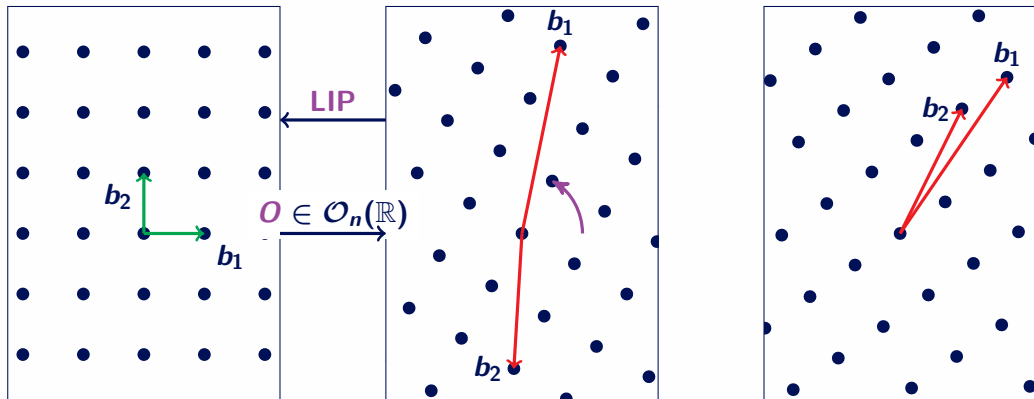


random lattice

decoding hard

Lattices and decoding

$$\text{Lattice } \mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$



integer lattice $\mathbb{Z}^n$

decoding easy

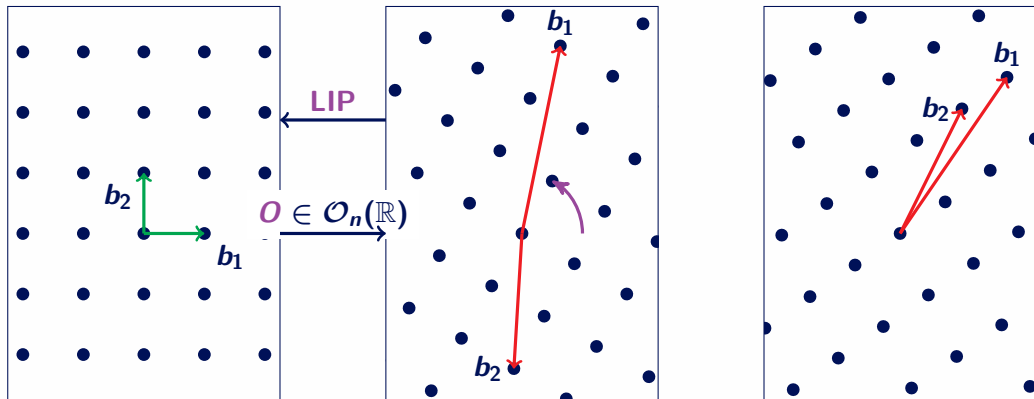
$\mathbb{Z}^n$ rotated

random lattice

decoding hard

Lattices and decoding

$$\text{Lattice } \mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$



integer lattice $\mathbb{Z}^n$

decoding easy

$\mathbb{Z}^n$ rotated

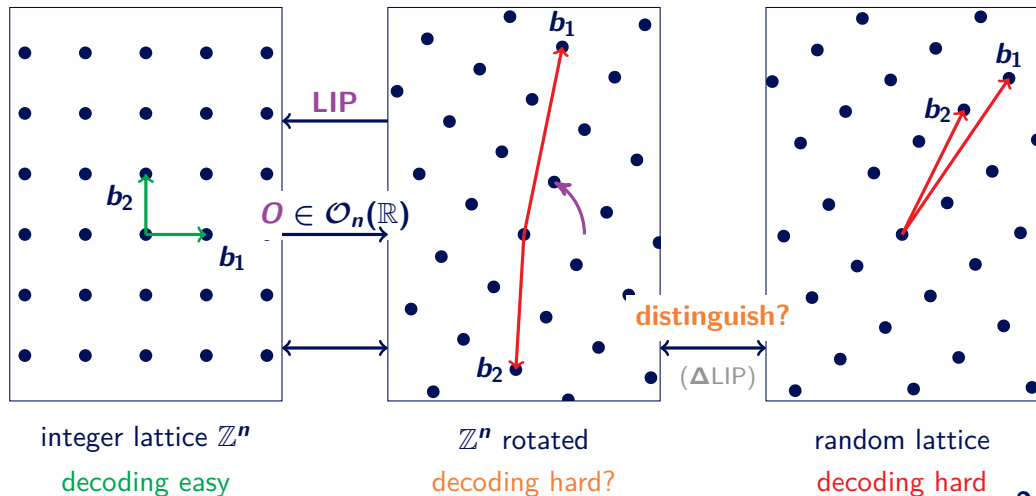
decoding hard?

random lattice

decoding hard

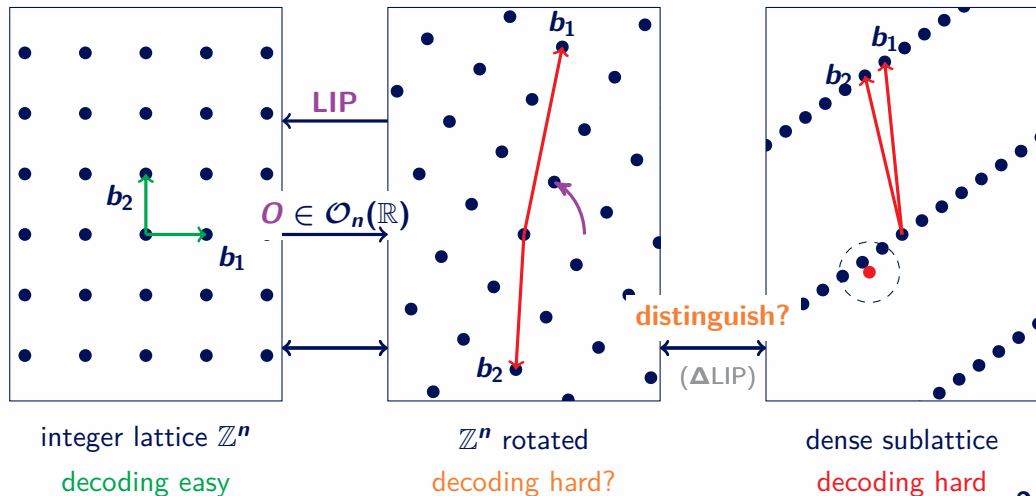
Lattices and decoding

$$\text{Lattice } \mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$



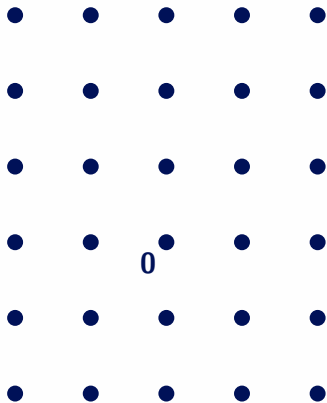
Lattices and decoding

$$\text{Lattice } \mathcal{L}(B) := \{\sum_i x_i b_i : x \in \mathbb{Z}^n\} \subset \mathbb{R}^n$$



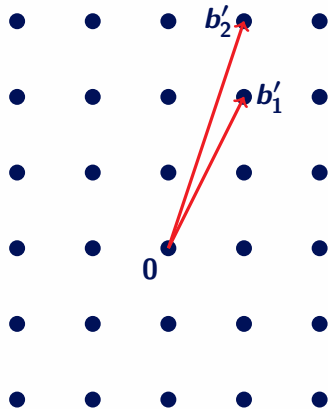
Encryption scheme based on LIP

Decodable lattice



$\mathcal{L}$

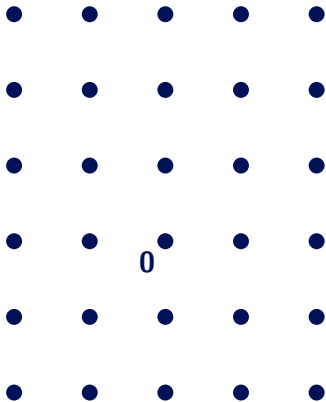
Bad basis of $\mathcal{L}$



$\mathcal{L}$

Encryption scheme based on LIP

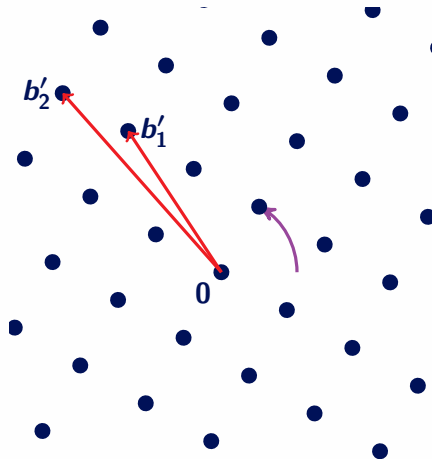
Decodable lattice



$\mathcal{L}$

$O \in \mathcal{O}_n(\mathbb{R})$
 $\xrightarrow{\text{(Secret key)}}$

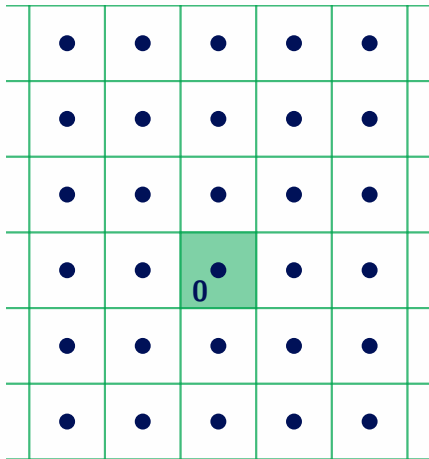
Bad basis of rotation



$O \cdot \mathcal{L}$

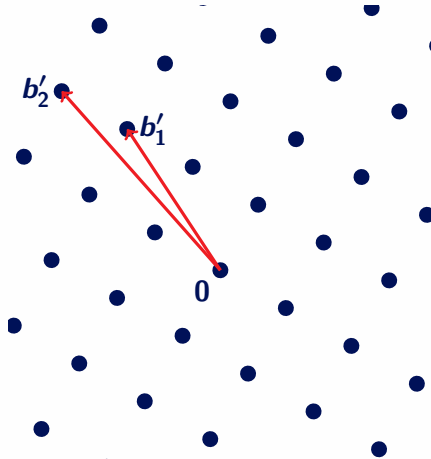
Encryption scheme based on LIP

Decodable lattice



$O \in \mathcal{O}_n(\mathbb{R})$
→
(Secret key)

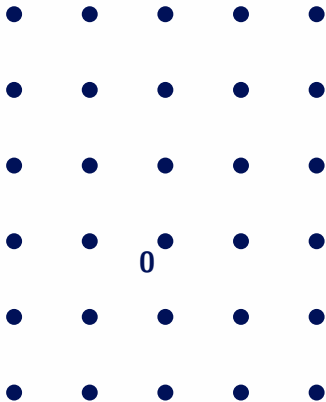
Bad basis of rotation



Hides (decoding) structure of $\mathcal{L}$

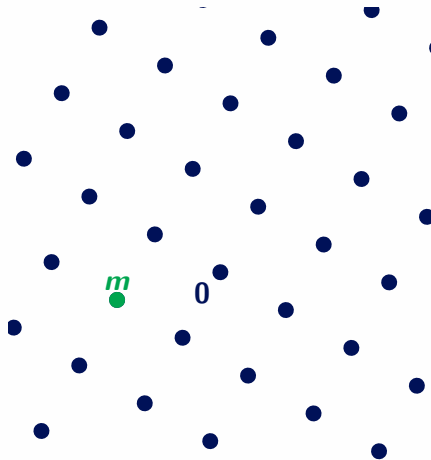
Encryption scheme based on LIP

Decodable lattice



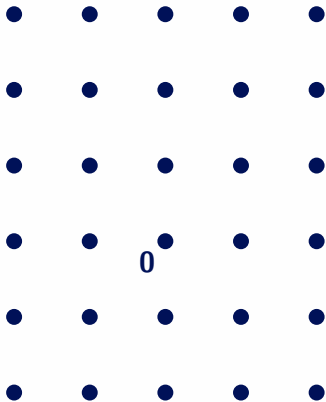
$$\begin{array}{c} \xrightarrow[\text{(Secret key)}]{O \in \mathcal{O}_n(\mathbb{R})} \end{array}$$

Bad basis of rotation



Encryption scheme based on LIP

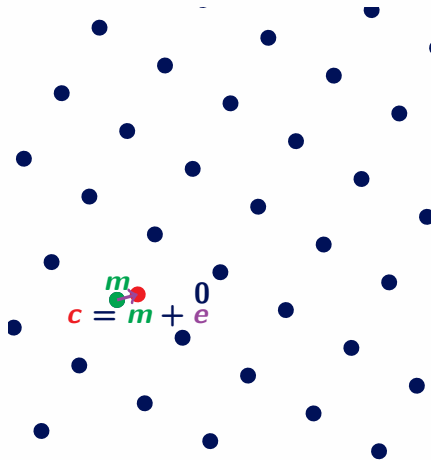
Decodable lattice



0

$$\xrightarrow[\text{(Secret key)}]{O \in \mathcal{O}_n(\mathbb{R})}$$

Bad basis of rotation

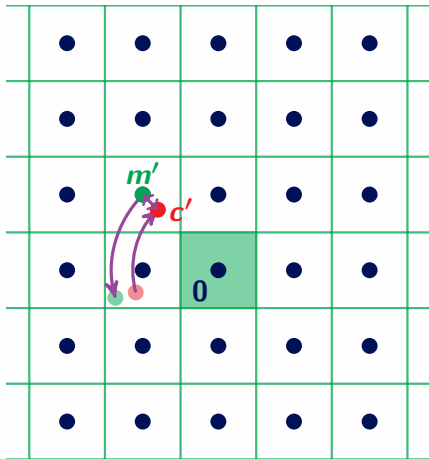


$$c = m + e$$

Encrypt by adding a small error

Encryption scheme based on LIP

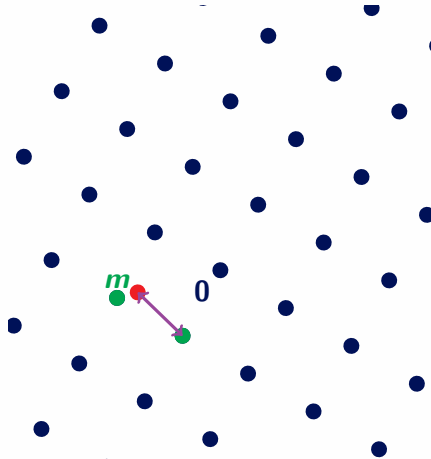
Decodable lattice



$$O \in \mathcal{O}_n(\mathbb{R})$$

(Secret key)

Bad basis of rotation



Decrypt using decoding algorithm

Lattice Isomorphism Problem

Definition: Lattice Isomorphism Problem (wc-sLIP)

Given $\mathbf{B}, \mathbf{B}' \in \text{GL}_n(\mathbb{R})$ of isomorphic lattices, find $\mathbf{O} \in \mathcal{O}_n(\mathbb{R})$ and $\mathbf{U} \in \text{GL}_n(\mathbb{Z})$
s.t. $\mathbf{B}' = \mathbf{O} \cdot \mathbf{B} \cdot \mathbf{U}$.

Lattice Isomorphism Problem

Definition: Lattice Isomorphism Problem (wc-sLIP)

Given $\mathbf{B}, \mathbf{B}' \in \text{GL}_n(\mathbb{R})$ of isomorphic lattices, find $\mathbf{O} \in \mathcal{O}_n(\mathbb{R})$ and $\mathbf{U} \in \text{GL}_n(\mathbb{Z})$
s.t. $\mathbf{B}' = \mathbf{O} \cdot \mathbf{B} \cdot \mathbf{U}$.

- ▶ Best known attacks require to **solve SVP**.

Lattice Isomorphism Problem

Definition: Lattice Isomorphism Problem (wc-sLIP)

Given $B, B' \in GL_n(\mathbb{R})$ of isomorphic lattices, find $O \in \mathcal{O}_n(\mathbb{R})$ and $U \in GL_n(\mathbb{Z})$ s.t. $B' = O \cdot B \cdot U$.

- ▶ Best known attacks require to **solve SVP**.
- ▶ Can be rephrased with Gram matrices $Q = B^\top B$, $Q' = (B')^\top B'$:

$$\text{find } U \in GL_n(\mathbb{Z}) \text{ s.t. } Q' = U^\top Q U$$

Lattice Isomorphism Problem

Definition: Lattice Isomorphism Problem (wc-sLIP)

Given $B, B' \in GL_n(\mathbb{R})$ of isomorphic lattices, find $O \in \mathcal{O}_n(\mathbb{R})$ and $U \in GL_n(\mathbb{Z})$ s.t. $B' = O \cdot B \cdot U$.

- ▶ Best known attacks require to **solve SVP**.
- ▶ Can be rephrased with Gram matrices $Q = B^\top B$, $Q' = (B')^\top B'$:

$$\text{find } U \in GL_n(\mathbb{Z}) \text{ s.t. } Q' = U^\top Q U$$

- ▶ Equivalence class $[Q] := \{U^\top Q U : U \in GL_n(\mathbb{Z})\}$.

average-case LIP and Δ LIP

- ▶ $\exists$ average-case distribution $\mathcal{D}_\sigma([Q])$ + efficient sampler (Q', U) (keygen!)

average-case LIP and Δ LIP

- ▶ $\exists$ average-case distribution $\mathcal{D}_\sigma([Q])$ + efficient sampler (Q', U) (keygen!)

Definition: **ac-sLIP** $_{\sigma}^Q$

Given $Q' \leftarrow \mathcal{D}_\sigma([Q])$, compute $U \in \text{GL}_n(\mathbb{Z})$ s.t. $Q' = U^T Q U$.

average-case LIP and Δ LIP

- ▶ $\exists$ average-case distribution $\mathcal{D}_\sigma([Q])$ + efficient sampler (Q', U) (keygen!)

Definition: **ac-sLIP** $_{\sigma}^Q$

Given $Q' \leftarrow \mathcal{D}_\sigma([Q])$, compute $U \in \text{GL}_n(\mathbb{Z})$ s.t. $Q' = U^T Q U$.

Definition: **ac- Δ LIP** $_{\sigma}^{Q_0, Q_1}$

Let $[Q_0] \neq [Q_1]$. Given $Q' \leftarrow \mathcal{D}_\sigma([Q_b])$, where $b \leftarrow \{0, 1\}$ uniform, recover b (with non-negligible advantage).

average-case LIP and Δ LIP

- ▶ $\exists$ average-case distribution $\mathcal{D}_\sigma([Q])$ + efficient sampler (Q', U) (keygen!)

Definition: **ac-sLIP** $_{\sigma}^Q$

Given $Q' \leftarrow \mathcal{D}_\sigma([Q])$, compute $U \in \text{GL}_n(\mathbb{Z})$ s.t. $Q' = U^T Q U$.

Definition: **ac- Δ LIP** $_{\sigma}^{Q_0, Q_1}$

Let $[Q_0] \neq [Q_1]$. Given $Q' \leftarrow \mathcal{D}_\sigma([Q_b])$, where $b \leftarrow \{0, 1\}$ uniform, recover b (with non-negligible advantage).

- ▶ $\sigma > 0$ determines the **quality of the basis**.
- ▶ For $\sigma > 2^{\Omega(n)} \cdot \lambda_1([Q])$ we have $\text{ac-sLIP}_{\sigma}^Q \geq \text{wc-sLIP}^Q$. (**WC-AC reduction**)

average-case LIP and Δ LIP

- ▶ $\exists$ average-case distribution $\mathcal{D}_\sigma([Q])$ + efficient sampler (Q', U) (keygen!)

Definition: **ac-sLIP** $_{\sigma}^Q$

Given $Q' \leftarrow \mathcal{D}_\sigma([Q])$, compute $U \in \text{GL}_n(\mathbb{Z})$ s.t. $Q' = U^T Q U$.

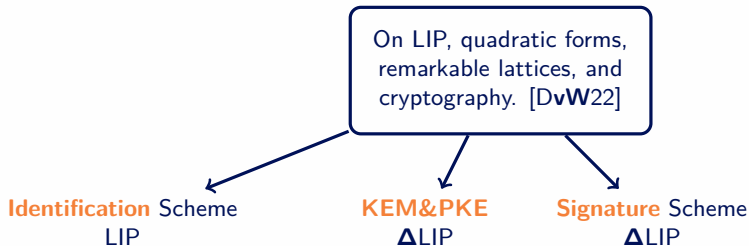
Definition: **ac- Δ LIP** $_{\sigma}^{Q_0, Q_1}$

Let $[Q_0] \neq [Q_1]$. Given $Q' \leftarrow \mathcal{D}_\sigma([Q_b])$, where $b \leftarrow \{0, 1\}$ uniform, recover b (with non-negligible advantage).

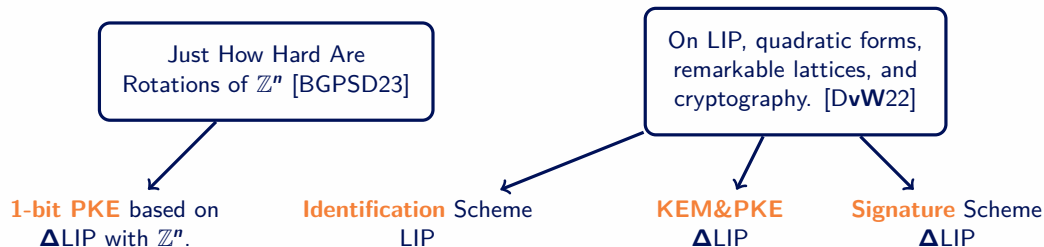
- ▶ $\sigma > 0$ determines the **quality of the basis**.
- ▶ For $\sigma > 2^{\Omega(n)} \cdot \lambda_1([Q])$ we have $\text{ac-sLIP}_{\sigma}^Q \geq \text{wc-sLIP}^Q$. (**WC-AC reduction**)

LIP and Δ LIP problems **depend on fixed lattice(s)**.

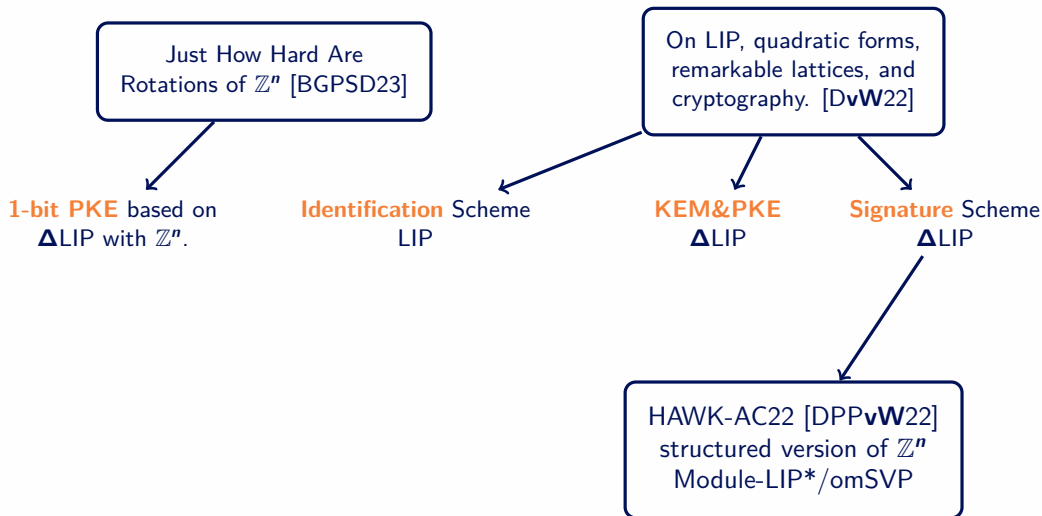
The beginnings



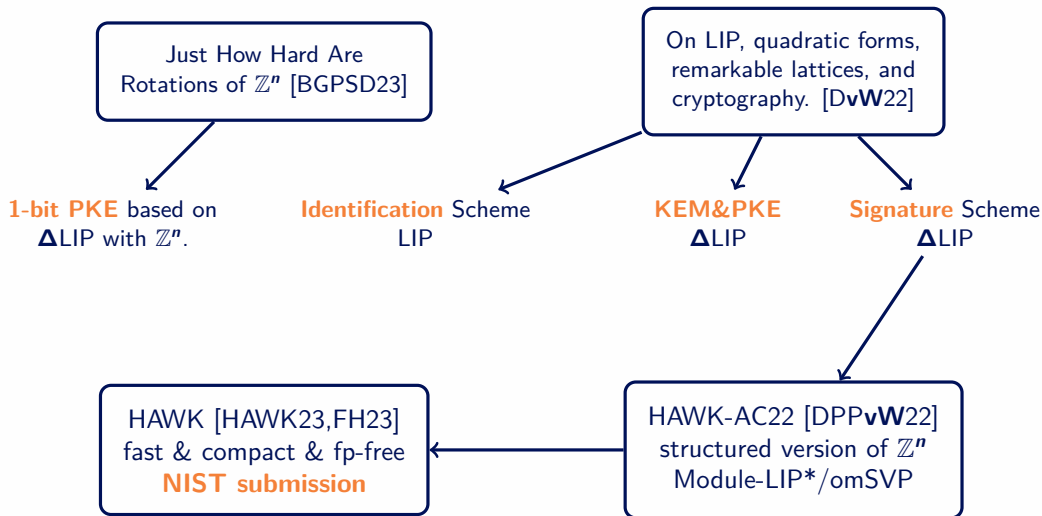
The beginnings



The beginnings



The beginnings



Constructions

KEM&PKEs

$\mathbb{Z}^n$ or BW_n + Δ LIP
[ARLW24,CBZIPC24]

IBE

Δ LIP
[BLSD26]

Various Commitment Schemes

LIP (group action)
[JWLLGPW25,LJPW25]

∞ -uPKE

Δ PCE [ABL25]
or Δ LIP (WIP)

Ring Signatures

LIP + HAWK
[LGD26]

Blind Signatures

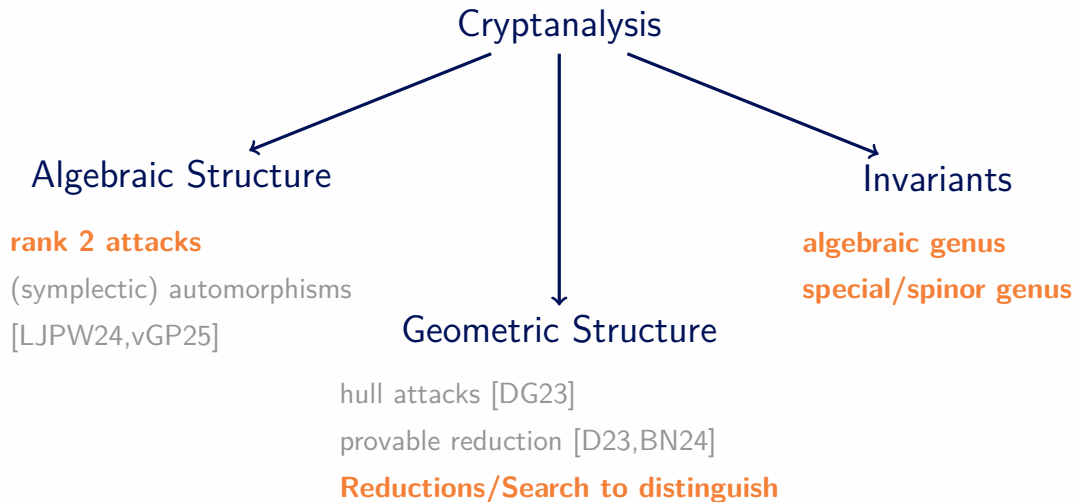
LIP (OR proofs)
[NP26]

(F)HE

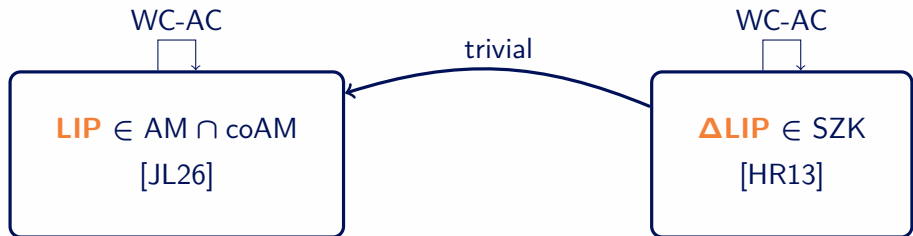
Δ LIP [BMM25,
LRvW25,BLSD26]

Allows for **Advanced Cryptographic Constructions**

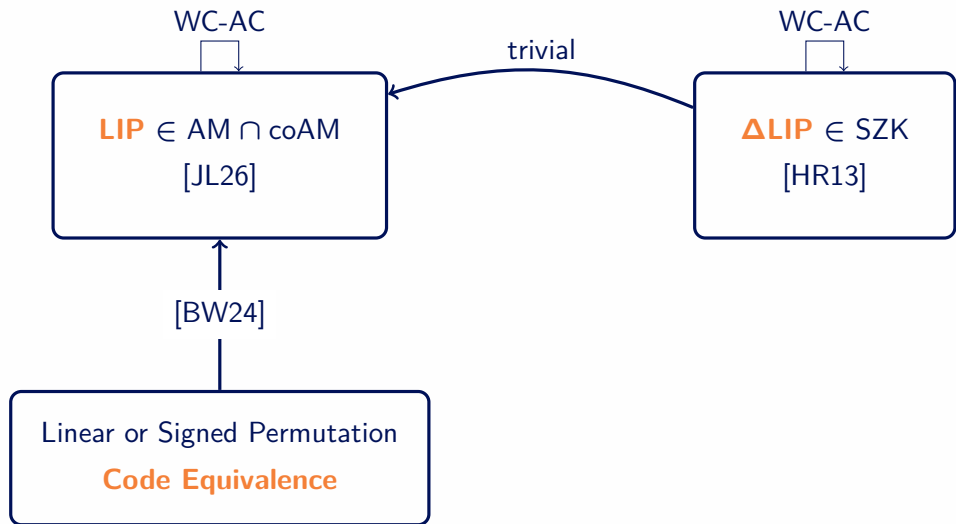
Cryptanalysis



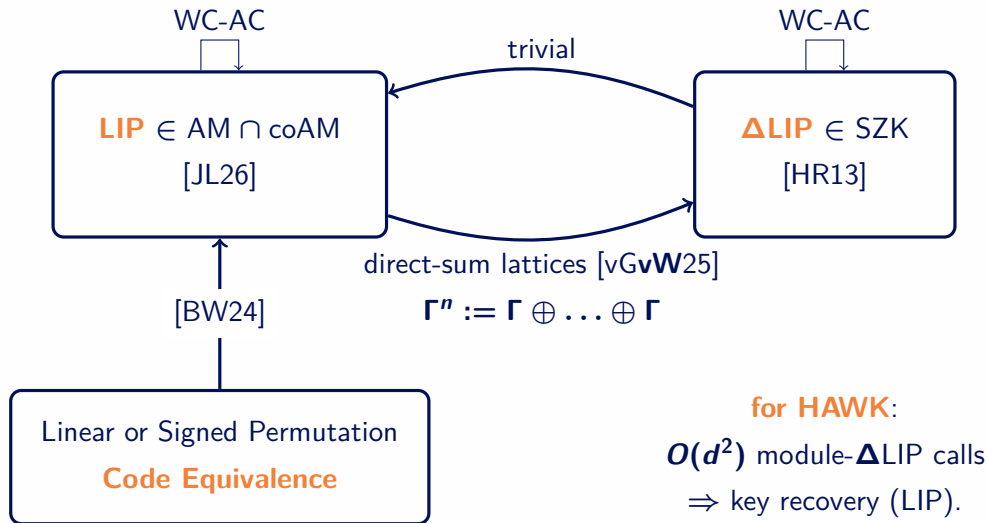
Reductions



Reductions



Reductions



Algebraic Structure

How much **structure** is allowed?

Algebraic Structure

How much **structure** is allowed?

Module-LIP: replace $\mathcal{L} = B \cdot \mathbb{Z}^n$ by $\mathcal{L} = B \cdot \mathcal{O}_K^r$ for ring of integers $\mathcal{O}_K$.

Algebraic Structure

How much **structure** is allowed?

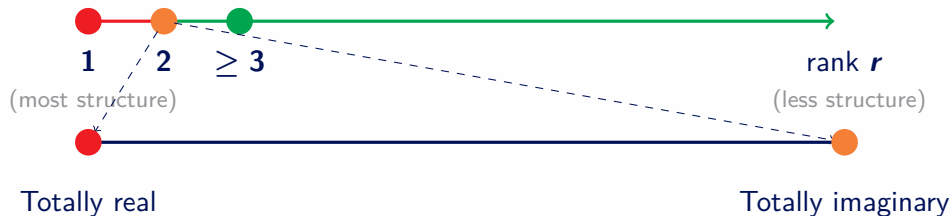
Module-LIP: replace $\mathcal{L} = B \cdot \mathbb{Z}^n$ by $\mathcal{L} = B \cdot \mathcal{O}_K^r$ for ring of integers $\mathcal{O}_K$.



Algebraic Structure

How much **structure** is allowed?

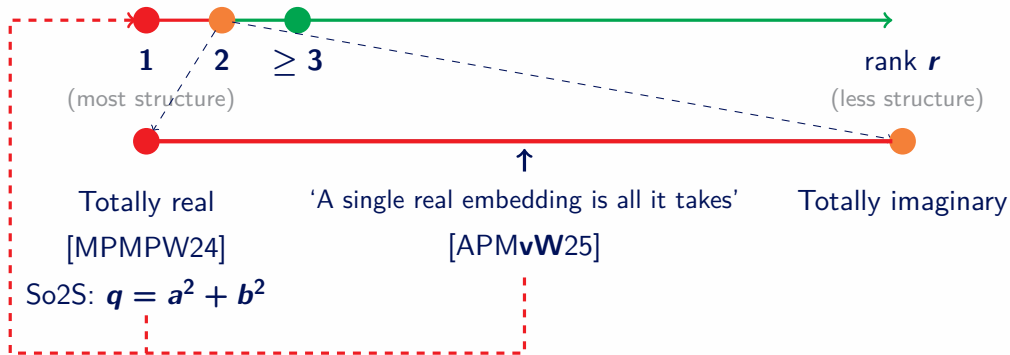
Module-LIP: replace $\mathcal{L} = B \cdot \mathbb{Z}^n$ by $\mathcal{L} = B \cdot \mathcal{O}_K^r$ for ring of integers $\mathcal{O}_K$.



Algebraic Structure

How much **structure** is allowed?

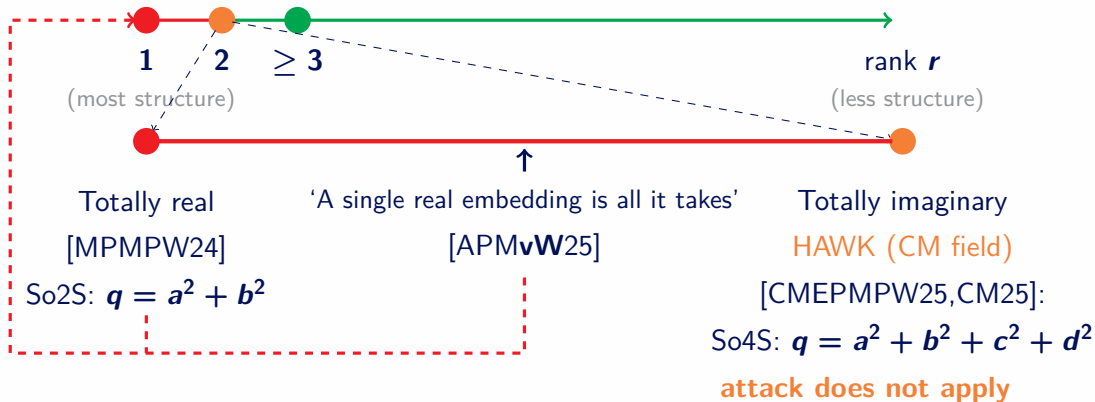
Module-LIP: replace $\mathcal{L} = B \cdot \mathbb{Z}^n$ by $\mathcal{L} = B \cdot \mathcal{O}_K^r$ for ring of integers $\mathcal{O}_K$.



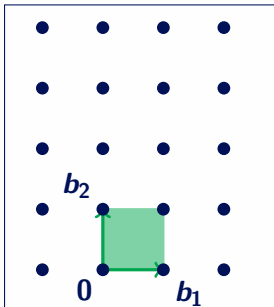
Algebraic Structure

How much **structure** is allowed?

Module-LIP: replace $\mathcal{L} = B \cdot \mathbb{Z}^n$ by $\mathcal{L} = B \cdot \mathcal{O}_K^r$ for ring of integers $\mathcal{O}_K$.

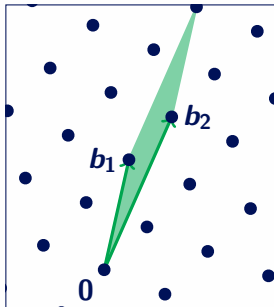


Invariants



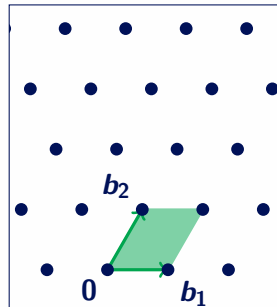
$\det(\mathcal{L}_1)$

$\stackrel{?}{=}$



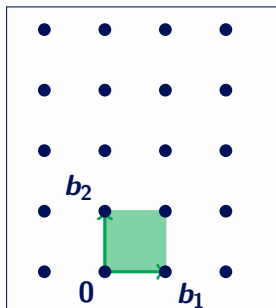
$\det(O \cdot \mathcal{L}_b)$

$\stackrel{?}{=}$



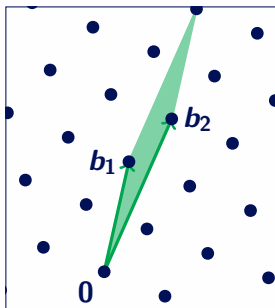
$\det(\mathcal{L}_2)$

Invariants



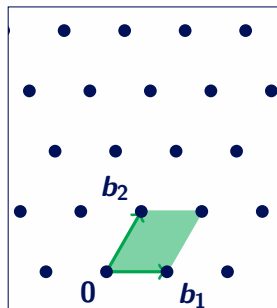
$\det(\mathcal{L}_1)$

$\stackrel{?}{=}$



$\det(O \cdot \mathcal{L}_b)$

$\stackrel{?}{=}$

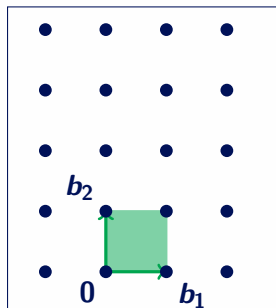


$\det(\mathcal{L}_2)$

Lemma:

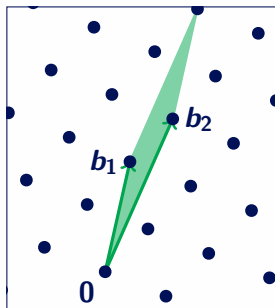
If $\det(\mathcal{L}_1) \neq \det(\mathcal{L}_2)$, then ΔLIP can be solved efficiently.

Invariants



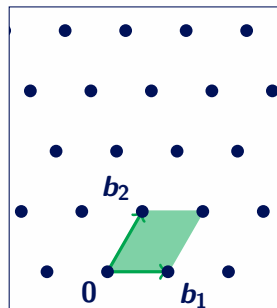
$\det(\mathcal{L}_1)$

$\stackrel{?}{=}$



$\det(O \cdot \mathcal{L}_b)$

$\stackrel{?}{=}$



$\det(\mathcal{L}_2)$

Lemma:

If $\det(\mathcal{L}_1) \neq \det(\mathcal{L}_2)$, then ΔLIP can be solved efficiently.

Conjecture [DvW22]: **Genus is the strongest** efficiently computable invariant.

Alternative LIP Assumptions

- ▶ Genus contains **finite #equivalence classes** and has a **natural distribution**.

Alternative LIP Assumptions

- Genus contains **finite #equivalence classes** and has a **natural distribution**.

[vW24] Random lattices in a genus behave like general random lattices.

$\Rightarrow$ any genus contains dense and smooth lattices $\Rightarrow$ tighter security proofs

Alternative LIP Assumptions

- ▶ Genus contains **finite #equivalence classes** and has a **natural distribution**.

[vW24] Random lattices in a genus behave like general random lattices.

$\Rightarrow$ any genus contains dense and smooth lattices $\Rightarrow$ tighter security proofs

Security Assumption: (informal) [LRvW25]

Let $\mathcal{L}$ be a lattice. For a random lattice $\mathcal{L}' \leftarrow \text{Genus}(\mathcal{L})$,
solving ΔBDD in $\mathcal{L}'$ is hard.

- ▶ Direct analogue of dLWE in family of random q -ary lattices.

Alternative LIP Assumptions

- ▶ Genus contains **finite #equivalence classes** and has a **natural distribution**.

[vW24] Random lattices in a genus behave like general random lattices.

$\Rightarrow$ any genus contains dense and smooth lattices $\Rightarrow$ tighter security proofs

Security Assumption: (informal) [LRvW25]

Let $\mathcal{L}$ be a lattice. For a random lattice $\mathcal{L}' \leftarrow \text{Genus}(\mathcal{L})$,
solving ΔBDD in $\mathcal{L}'$ is hard.

- ▶ Direct analogue of dLWE in family of random q -ary lattices.
- ▶ 'vintage' McEliece: **distinguish assumption** + **random lattice problem**.

Alternative LIP Assumptions

- ▶ Genus contains **finite #equivalence classes** and has a **natural distribution**.

[vW24] Random lattices in a genus behave like general random lattices.

$\Rightarrow$ any genus contains dense and smooth lattices $\Rightarrow$ tighter security proofs

Security Assumption: (informal) [LRvW25]

Let $\mathcal{L}$ be a lattice. For a random lattice $\mathcal{L}' \leftarrow \text{Genus}(\mathcal{L})$,
solving ΔBDD in $\mathcal{L}'$ is hard.

- ▶ Direct analogue of dLWE in family of random q -ary lattices.
- ▶ ‘vintage’ McEliece: **distinguish assumption** + **random lattice problem**.
- ▶ (WIP) **WC-AC reduction** within a fixed genus for BDD, CVP, DGS, ...

LIP as a group action

- ▶ (Right) **group action** on equivalence class $[Q := B^T B]$ by $U \in \text{GL}_d(\mathbb{Z})$:

$$U \circ Q \mapsto U^T Q U$$

LIP as a group action

- ▶ (Right) **group action** on equivalence class $[Q := B^T B]$ by $U \in \text{GL}_d(\mathbb{Z})$:

$$U \circ Q \mapsto U^T Q U$$

- ▶ Group is **infinite** (not even compact!) and **non-commutative**.
 $\Rightarrow$ Can't sample 'uniformly' from $\mathbf{G} = \text{GL}_d(\mathbb{Z})$.

LIP as a group action

- ▶ (Right) **group action** on equivalence class $[Q := B^T B]$ by $U \in \text{GL}_d(\mathbb{Z})$:

$$U \circ Q \mapsto U^T Q U$$

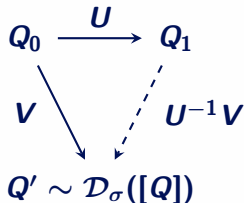
- ▶ Group is **infinite** (not even compact!) and **non-commutative**.
 $\Rightarrow$ Can't sample 'uniformly' from $G = \text{GL}_d(\mathbb{Z})$.
- ▶ Instead: sample image $Q' \leftarrow \mathcal{D}_\sigma([Q])$ **along with** uniform $U \in \text{Isom}(Q, Q')$.

LIP as a group action

- ▶ (Right) **group action** on equivalence class $[Q := B^T B]$ by $U \in \text{GL}_d(\mathbb{Z})$:

$$U \circ Q \mapsto U^T Q U$$

- ▶ Group is **infinite** (not even compact!) and **non-commutative**.
 $\Rightarrow$ Can't sample 'uniformly' from $\mathbf{G} = \text{GL}_d(\mathbb{Z})$.
- ▶ Instead: sample image $Q' \leftarrow \mathcal{D}_\sigma([Q])$ **along with** uniform $U \in \text{Isom}(Q, Q')$.
- ▶ ZKPoK [DvW21]:



$$(Q', V) \leftarrow \text{Sample}_\sigma(Q_0)$$

for $\sigma^2 \geq \tilde{\Omega}(\max\{\|\tilde{Q}_0\|, \|\tilde{Q}_1\|\})$

Mistakes to avoid

 LIP does not give a standard cryptographic group action.

Mistakes to avoid

⚠ LIP does not give a standard cryptographic group action.

- ▶ Alternatives to $(Q_1, U) \leftarrow \text{Sample}_\sigma(Q_0)$ are typically **insecure**.

Mistakes to avoid

⚠ LIP does not give a standard cryptographic group action.

- ▶ Alternatives to $(Q_1, U) \leftarrow \text{Sample}_\sigma(Q_0)$ are typically **insecure**.

[BM21] **Never sample $U \in \text{GL}_d(\mathbb{Z})$ first** (independent of Q_0).

Mistakes to avoid

⚠ LIP does not give a standard cryptographic group action.

- ▶ Alternatives to $(Q_1, U) \leftarrow \text{Sample}_\sigma(Q_0)$ are typically **insecure**.

[BM21] **Never sample $U \in \text{GL}_d(\mathbb{Z})$ first** (independent of Q_0).

[BBCK24,BCDF25] **Never (re)use $U \in \text{GL}_d(\mathbb{Z})$ on another Q_2 .** (~~WP~~)

Mistakes to avoid

⚠ LIP does not give a standard cryptographic group action.

- ▶ Alternatives to $(Q_1, U) \leftarrow \text{Sample}_\sigma(Q_0)$ are typically **insecure**.

[BM21] **Never sample** $U \in \text{GL}_d(\mathbb{Z})$ **first** (independent of Q_0).

[BBCK24,BCDF25] **Never (re)use** $U \in \text{GL}_d(\mathbb{Z})$ on another Q_2 . (~~WP~~)

Led to several **broken** constructions in group-action-based frameworks.

- ▶ See [NP26] for a more careful treaty of LIP as a group action.

The Future of LIP?

The Future of LIP?

Better lattices $\Rightarrow$ smaller keys and ciphertexts

The Future of LIP?

Better lattices $\Rightarrow$ smaller keys and ciphertexts

Lattices with unique algorithmic properties?

The Future of LIP?

Better lattices $\Rightarrow$ smaller keys and ciphertexts

Lattices with unique algorithmic properties?

Reductions: $WC \rightarrow AC$ within any genus? (WIP)

$LIP \leftrightarrow \Delta LIP?$

The Future of LIP?

Better lattices $\Rightarrow$ smaller keys and ciphertexts

Lattices with unique algorithmic properties?

Reductions: $WC \rightarrow AC$ within any genus? (WIP)

$LIP \leftrightarrow \Delta LIP?$

Cryptanalysis: module-LIP,
stronger invariants & remarkable lattices

The Future of LIP?

Better lattices $\Rightarrow$ smaller keys and ciphertexts

Lattices with unique algorithmic properties?

Reductions: $WC \rightarrow AC$ within any genus? (WIP)

$LIP \leftrightarrow \Delta LIP?$

Cryptanalysis: module-LIP,
stronger invariants & remarkable lattices

Thank you!

Bibliography (1/3)

- [ABL25] Hollow LWE: A New Spin: Unbounded Updatable Encryption from LWE and PCE
- [APMvW25] cryptanalysis of rank-2 module-LIP: a single real embedding is all it takes
- [ARLW24] Public-key encryption from the lattice isomorphism problem
- [HAWK23] HAWK <https://hawk-sign.info/>
- [BDG23] Genus distribution of random q -ary lattices
- [BGPSD23] Just How Hard Are Rotations of $\mathbb{Z}^n$? Algorithms and Cryptography with the Simplest Lattice
- [BMM25] Fully-Homomorphic Encryption from Lattice Isomorphism
- [BN24] Improved provable reduction of NTRU and hypercubic lattices
- [BW25] Relating code equivalence to other isomorphism problems
- [CBZIPC24] A concrete LIP-based KEM with simple lattices
- [BLSD26] Advanced cryptography from lattice isomorphism—new constructions of IBE and FHE

Bibliography (2/3)

- [CM25] Ideally HAWKward: How Not to Break Module-LIP
- [CMEPMPW25] A reduction from Hawk to the principal ideal problem in a quaternion algebra
- [D23] Provable lattice reduction of with blocksize $n/2$
- [DG23] Hull attacks on the lattice isomorphism problem
- [DPPvW22] Hawk: Module LIP Makes Lattice Signatures Fast, Compact and Simple
- [DvW22] On the lattice isomorphism problem, quadratic forms, remarkable lattices, and cryptography
- [FH23] On the Quantum Security of HAWK
- [JWLLGPW25] Re-randomize and extract: A novel commitment construction framework based on group actions
- [LJPW24] Cryptanalysis of rank-2 module-LIP with symplectic automorphisms
- [LGD26] RingSLIP: Ring Signatures from the Lattice Isomorphism Problem
- [JL26] Exploiting the complexity of Lattice Isomorphism Problem via Irreducible Decomposition

Bibliography (3/3)

- [LJPW25] Commitment Schemes Based on Module-LIP
- [LLM24] On the spinor genus and the distinguishing lattice isomorphism problem
- [LRvW25] Beyond LWE: a Lattice Framework for Homomorphic Encryption
- [M25] Special Genera of Hermitian Lattices and Applications to HAWK
- [MPMPW24] Cryptanalysis of rank-2 module-lip in totally real number fields
- [vG25] A note on the genus of the HAWK lattice
- [vGP25] HAWK: Having Automorphisms Weakens Key
- [vGvW25] A search to distinguish reduction for the isomorphism problem on direct sum lattices
- [vW24] Dense and smooth lattices in any genus
- [KS26] (Mis)using the Lattice Isomorphism Problem Cryptanalysis of the double-LIP and Construction of LIP-Based Blind Signatures
- [NP26] Post-Quantum Anonymous Signatures from the Lattice Isomorphism Group Action