

Weinstein's category for shifted symplectic structures via groupoids

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Under the motto that “everything is Lagrangian”, Alan Weinstein proposed a category whose objects are symplectic manifolds and whose morphisms are Lagrangian correspondences. As Poisson geometry has developed over the past decades—motivated largely by classical mechanics—additional structures such as Poisson, Dirac, and Courant have emerged. Safronov, drawing on the shifted symplectic structures of Pantev–Toën–Vaquié–Vezzosi (PTVV), places these geometric structures into a unified framework by viewing them as symplectic structures of various shifts. Calaque and Scheinbauer constructed a Weinstein category for shifted symplectic structures using higher derived stacks in the sense of Toën–Vezzosi. While the language of stacks is intrinsic, it remains rather implicit for differential geometers and therefore difficult to use for concrete calculations. Pridham approaches higher derived stacks via presentations by groupoids, a setting far more familiar to researchers in differential geometry, dynamical systems, and noncommutative geometry. Inspired by Pridham’s approach, we construct a Weinstein category for shifted symplectic structures using higher derived Lie groupoids. However, the topology we use for derived manifolds differs from that of Toën–Vezzosi and Pridham: we work with fibrations admitting local sections, which allow us to treat the odd line and cotangent groupoids—key examples in Poisson geometry.

We prove that when intersections are transversal in a higher derived sense, the composition of Lagrangian correspondences remains Lagrangian. As always, there is technical difficulty when intersection is not transversal. Broadly, two methods are available: (1) perturbation, as in the Fukaya category, which yields explicit results but requires delicate analysis; or (2) (fibrant) replacement, as in PTVV, which is conceptual but requires a homotopical framework and typically leaves explicit fibrant replacements to be worked out.

We adopt the second method and build an iCFO (incomplete category of fibrant objects) for derived higher Lie groupoids and provide explicit fibrant replacements using a collection of tubular neighborhood theorems: Weinstein’s Lagrangian tubular neighborhood theorem, and the Hoyo–Fernandes version for Lie groupoids via the Crainic–Fernandes–Torres PMCT program. As applications, we use Calaque–Safronov’s trick, then singular symplectic reduction, quasi-symplectic reduction, and Lu–Weinstein reduction, all appear as shifted symplectic derived Lie groupoids. This is based on a joint work with Miquel Cueva, Florian Dorsch and Reyer Sjamaar.