Solving Hamilton-Jacobi equations by residual minimization of monotone schemes

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We introduce a method for solving Hamilton--Jacobi equations, both inviscid and viscous, by minimizing the squared residuals of monotone finite-difference discretizations on grids of varying resolution. The method is designed to leverage neural networks and modern GPUs to solve these equations in higher dimensions; consequently, the setting for our analysis is the minimization of the residual functionals via gradient-based optimization. We establish a well-posedness theory for this approach: any critical point of the finite-difference loss solves the monotone scheme together with the prescribed Dirichlet boundary conditions, and the error of an approximation is controlled by its residual. We then derive the rate of convergence of the gradient flow that minimizes the residual in several settings, noting that the rate may depend on the grid resolution and the domain dimension. Building on this foundation, we propose a multi-level training algorithm that exploits the faster convergence available on the coarser grids of the discretization. Combined with the convergence theorem of Barles and Souganidis for monotone and consistent schemes, our results guarantee convergence to the unique viscosity solution as the grid is refined. We illustrate the approach on eikonal equations, level-set problems, and a Hamilton--Jacobi--Isaacs equation arising from a stochastic differential game, in dimensions up to eight.