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Expanding the definition of a finite element: groups, complexes and software

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Oxford Mathematics

Partial differential equations represent complex problems

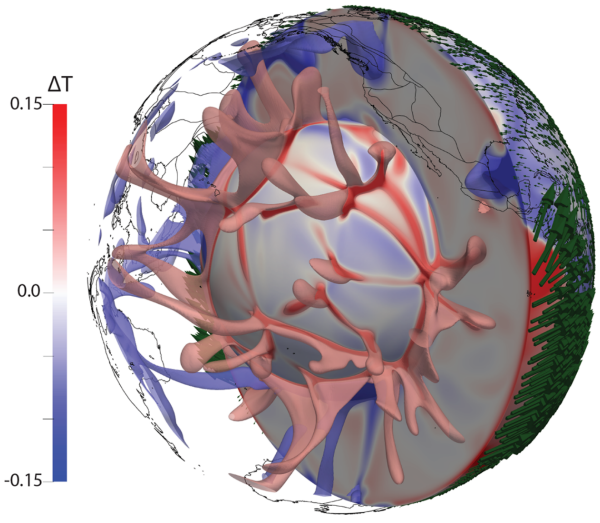

Figure: Gusto team, Australian National University

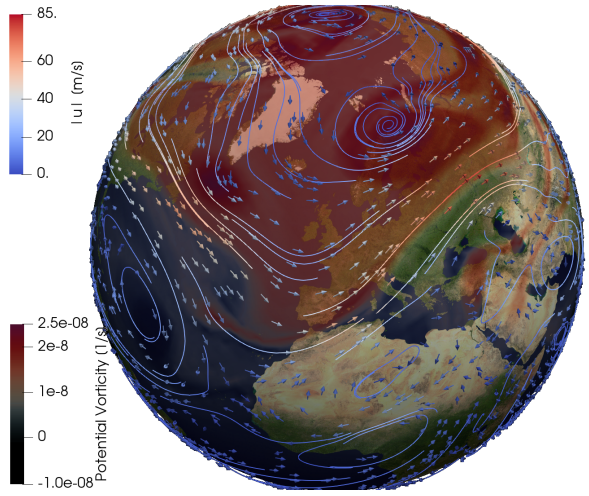

Figure: Gusto team, Exeter University, Met Office

Breaking down the problem

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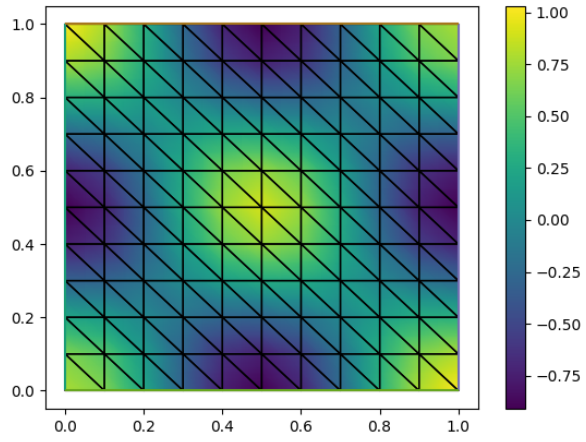

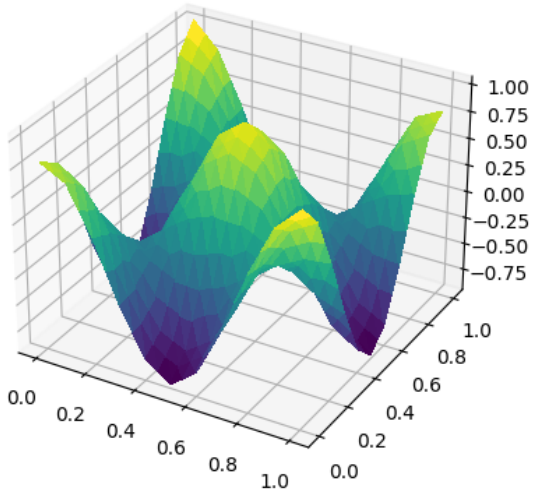

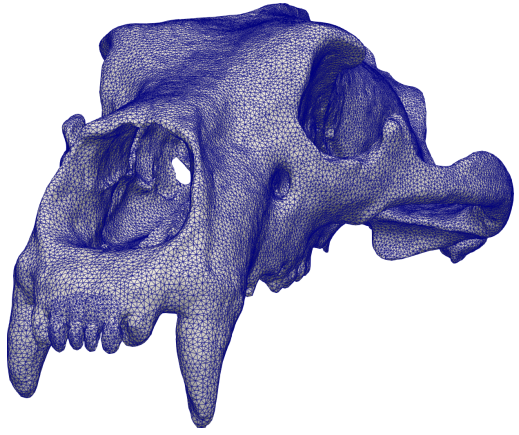

Outline of the finite element method

1. Choose a PDE + boundary conditions

$$u - \nabla^2 u = f \qquad\qquad \nabla u \cdot \hat{n} = 0 \text{ on the boundary } \Gamma$$

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4. Construct and solve a linear system, using a basis ϕ_i for V_h, and seeking $u_h = \sum_i u_i \phi_i \in V_h$:

$$Ku = f \qquad u^T = (u_0, u_1, \dots)$$

$$K_{ij} = \int_\Omega \phi_i \phi_j + \nabla \phi_i \nabla \phi_j \mathrm{d}\, x \qquad f_i = \int_\Omega \phi_i \mathrm{d}\, x$$

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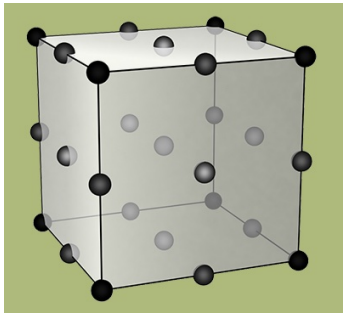
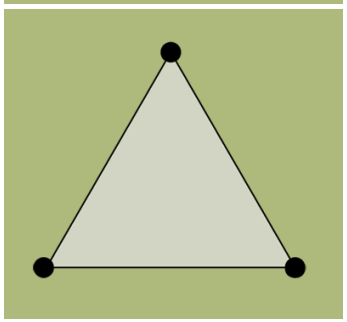
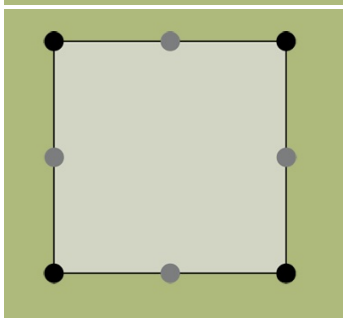
Choosing an element



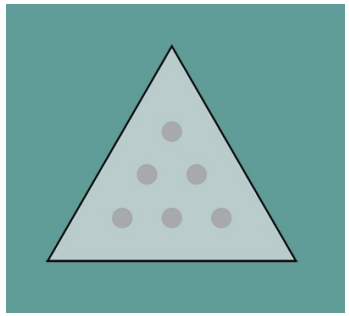


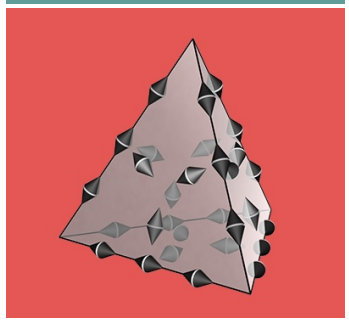


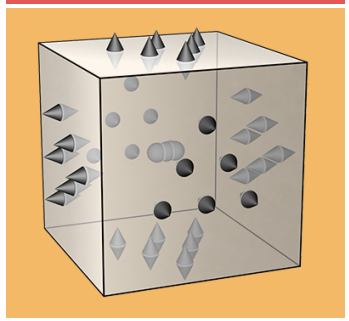
There are many factors that influence the choice of element:

- Degree of approximation
- Cell geometry
- Choice of element pairs for stability
- Function space being approximated

Diagrams: D. N. Arnold and A. Logg, Periodic Table of the Finite Elements, SIAM News, vol. 47 no. 9, November 2014.







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Ciarlet's classical definition of a finite element is $(\mathcal{K}, \mathcal{V}, \mathcal{L})$ where

- $\mathcal{K}$ is the cell, a bounded closed subset of $\mathbb{R}^n$ with nonempty interior and piecewise smooth boundary;
- $\mathcal{V}$ is a suitable finite dimensional space on $\mathcal{K}$;
- $\mathcal{L}$ is the set of degrees of freedom $\mathcal{L} = \{\ell_1, \ldots \ell_n\}$, a basis for $\mathcal{V}^*$.

This has been fundamental to the success of the finite element method.

What is Ciarlet's definition missing?

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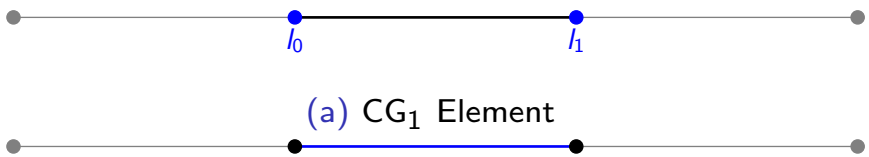
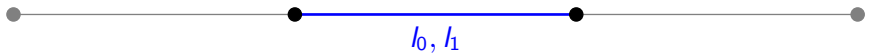
- Cell structure (geometry, topology)
- Association of degrees of freedom to topological entities
- Relationships between degrees of freedom (symmetry)



(a) CG_1 Element



(b) DG_1 Element

Each software makes its own assumptions to patch the missing information from Ciarlet.

A new definition

Our new definition is $(\mathcal{C}, \mathcal{V}, \mathcal{E})$ where:

- $\mathcal{C}$ is a cell complex, annotated with orientation information;
- $\mathcal{V}$ is a triple of spaces: the polynomial space on the cell, the space being approximated and the space required for interpolation of the element;
- $\mathcal{E}$ is a description of how to *generate* the degrees of freedom, $\mathcal{L}$.

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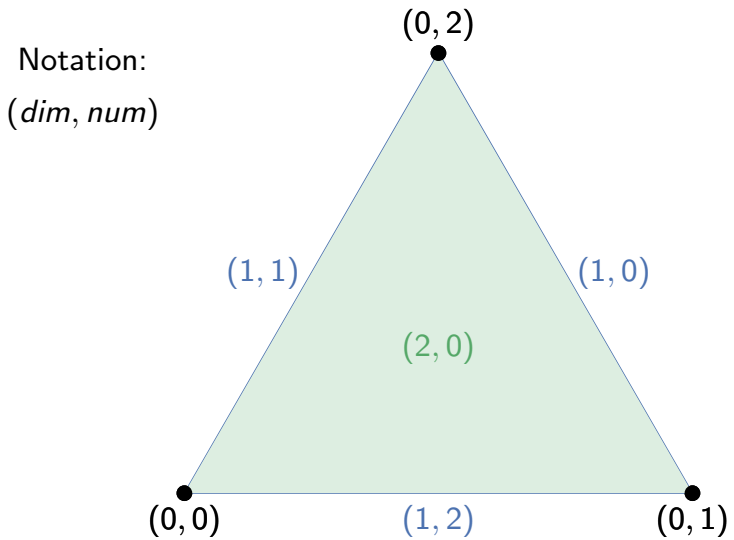
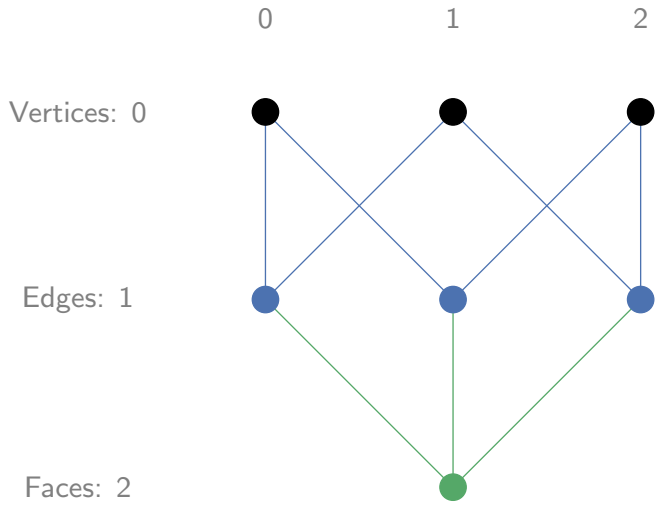
$\mathcal{E}$ is composed of three elements, $(\mathcal{X}, \mathcal{G}_1, \mathcal{G}_2)$:

- $\mathcal{X}$ is a set of generators for the degrees of freedom, parameterised by a group member $g \in \mathcal{G}_1$;
- $\mathcal{G}_1$ is the generator group, from which we draw g;
- $\mathcal{G}_2$ is the transformation group, which describes how the values of the degrees of freedom transform under the symmetry group of the cell.

The cell is now defined using a **cell complex**, which we define as a set of points with dimension and connections between those points.





Annotating the connections with orientation information incorporates orientation assumptions made about the cell.

CG1 in the new definition

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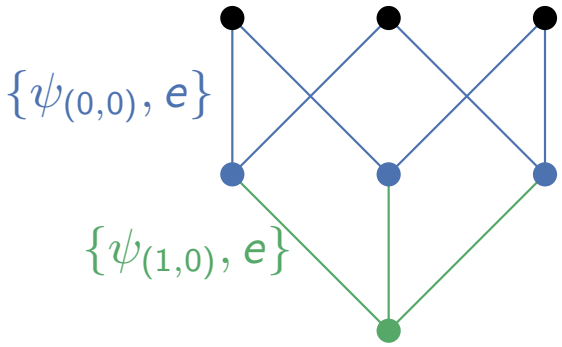
$\mathcal{C}_{\mathsf{CG}_1} =$



$$\psi_{(0,0)} := \{() \to (-1)\}, \psi_{(0,1)} := \{() \to (1)\}$$

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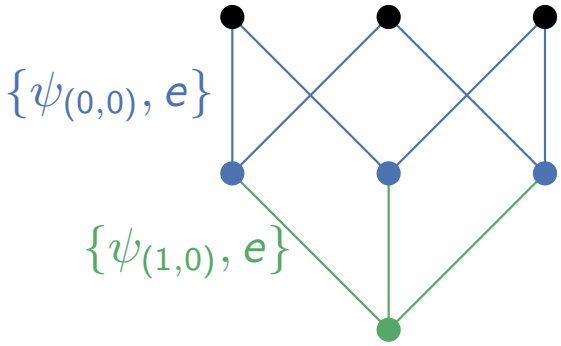

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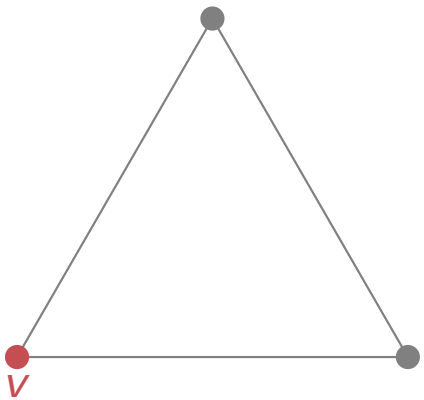

$$\mathcal{V}_{\mathsf{CG}_1} = (P_1(\bar{\mathcal{C}}), H^1(\mathcal{C}), C^0)$$

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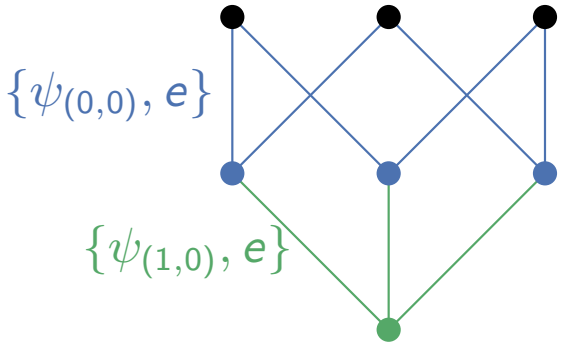

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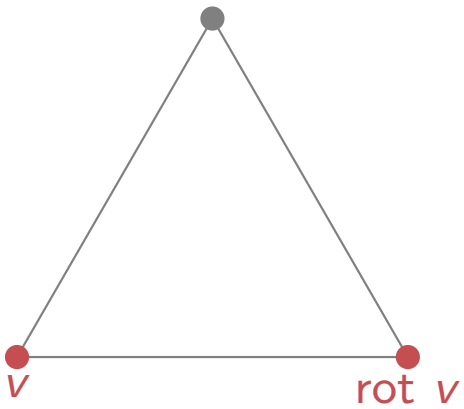

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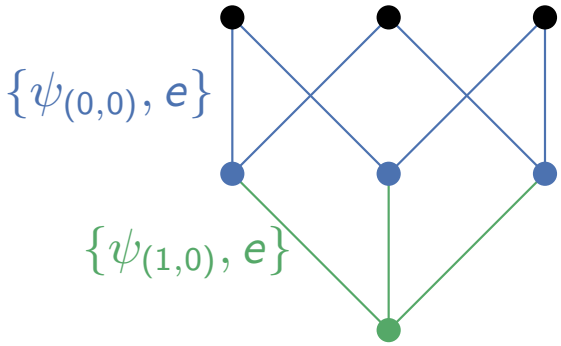

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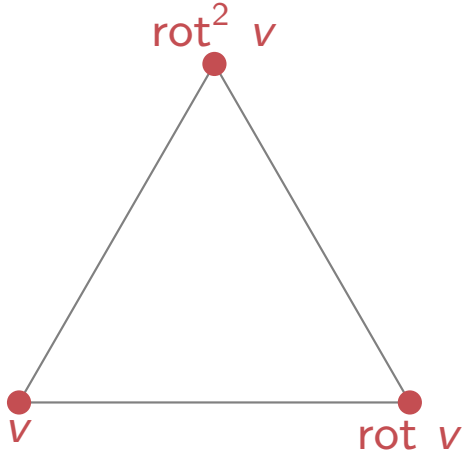

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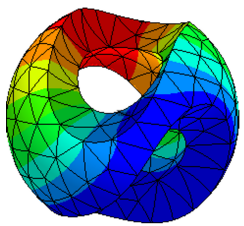
Most finite element software allows the user to choose from a set of hard coded elements. In order to use a new element, it needs to be implemented, typically by a developer of the software.

Firedrake

FENICS PROJECT

deal.II



NGSolve

FUSE

FUSE is an open source Python package providing a domain specific language that implements this definition.

With the new definition, we have all the information any finite element software needs to implement an element.

It should be possible to adapt any software to directly use elements written by users in FUSE without the need for internal code modification.

`https://github.com/firedrakeproject/fuse`
(under construction!!)

Example

CG1 on a triangle can be constructed and then used in Firedrake using FUSE and the new definition with this program:

```
1   from firedrake import *
2   from fuse import *
3   cell = polygon(3)
4   vert = cell.vertices()[0]
5
6   xs = [DOF(DeltaPairing(), PointKernel(()))]
7   V = (P0, CellL2, C0)
8   E = DOFGenerator(xs, S1, S1)
9   dg0 = ElementTriple(vert, V, E)
10
11  xs = [immerse(cell, dg0, TrH1)]
12  V = (P1, CellH1, C0)
13  E =  DOFGenerator(xs, S3 / S2, S1)
14  cg1 = ElementTriple(cell, V, E)
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16  mesh = UnitSquareMesh(10, 10)

17  V = FunctionSpace(mesh,   cg1.to_ufl()  )
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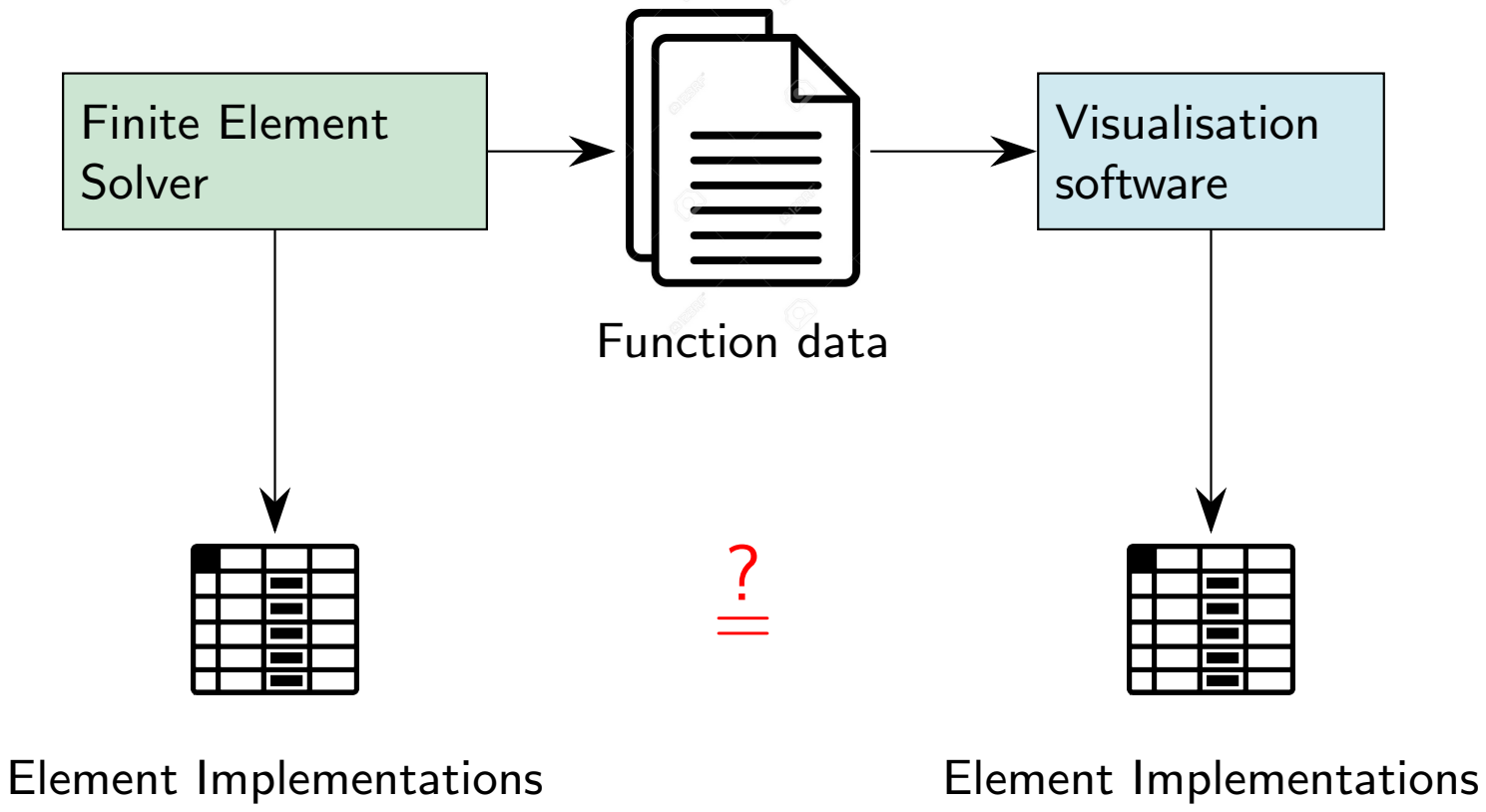
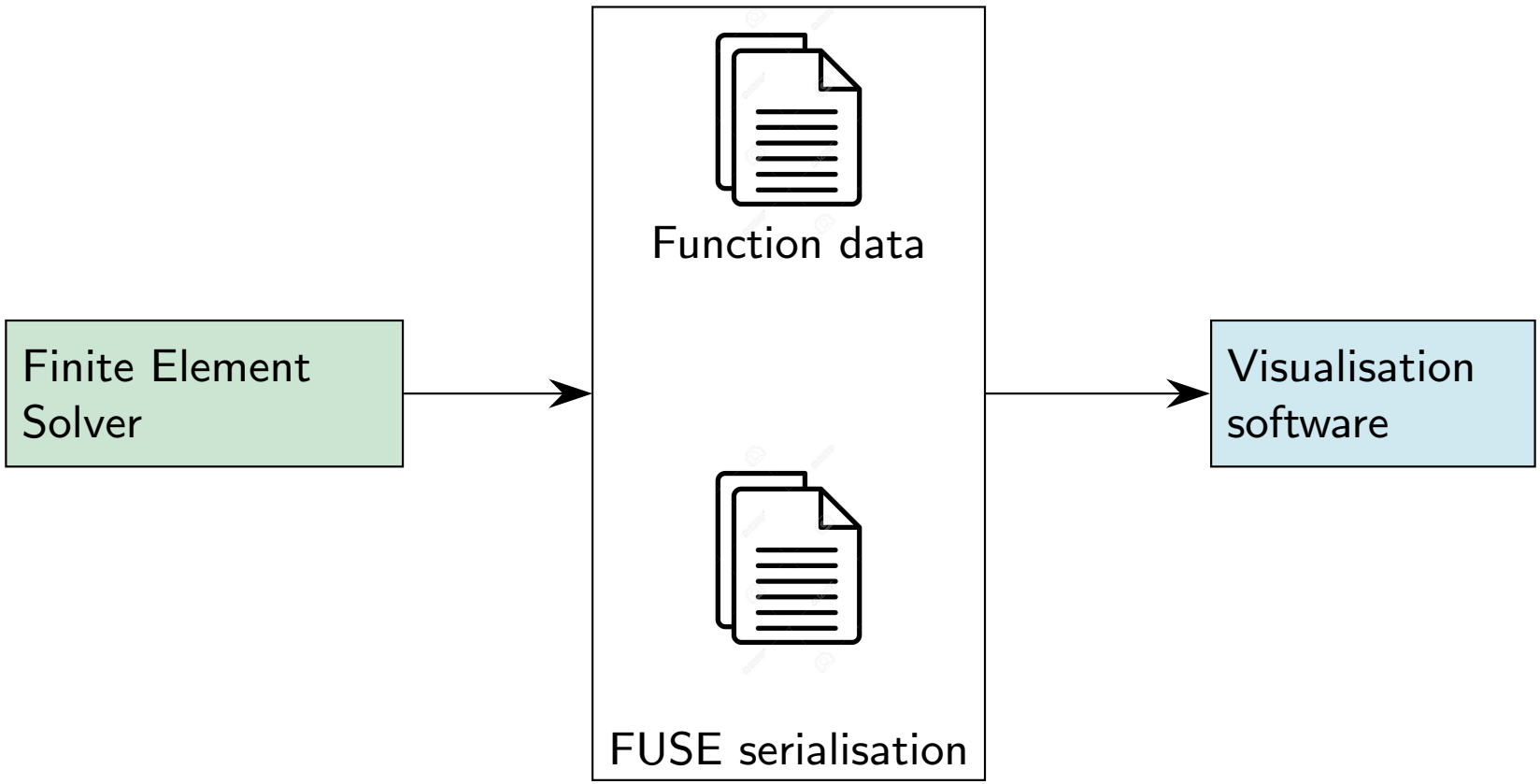
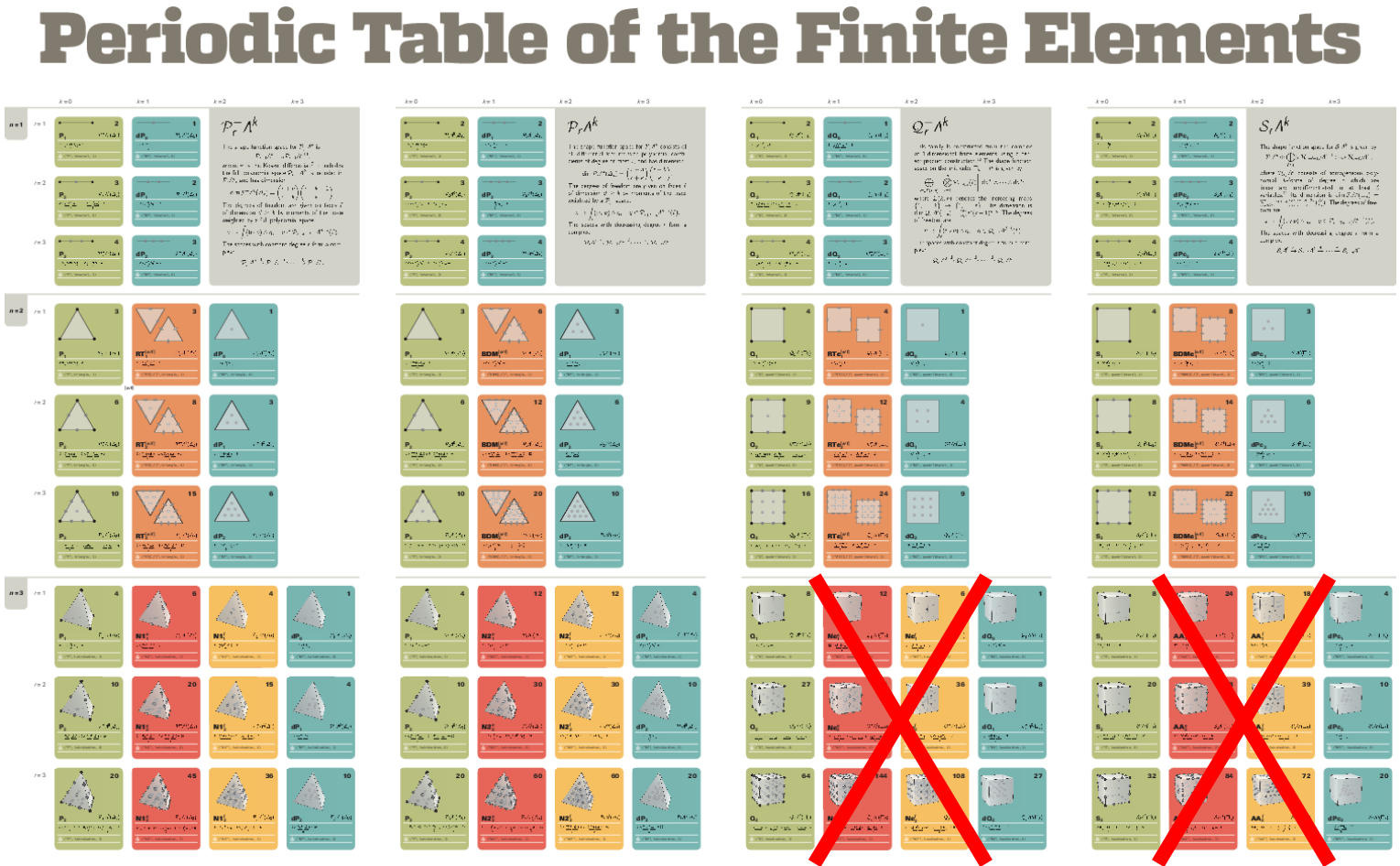
Communicating with other software



Communication with other software - with Fuse serialisation



Extension of support for periodic table



D. N. Arnold and A. Logg, Periodic Table of the Finite Elements, SIAM News, vol. 47 no. 9, November 2014.

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- **Exploitation of new structure in Firedrake** - Orientation information, non standard pullbacks.
- **Integration of FUSE with other software (eg VTK)**
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Thank you for listening!

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