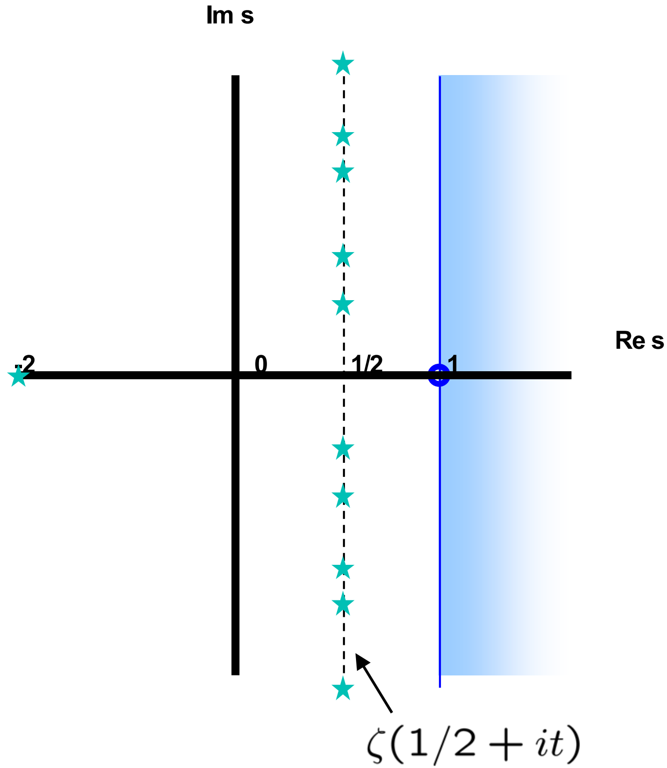
University of BRISTOL

Hollywood’s hippest mathematics: Random matrix theory and the Riemann zeta function

January 12th, 2026

Nina Snaith

Riemann zeta function



For Re$s > 1$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

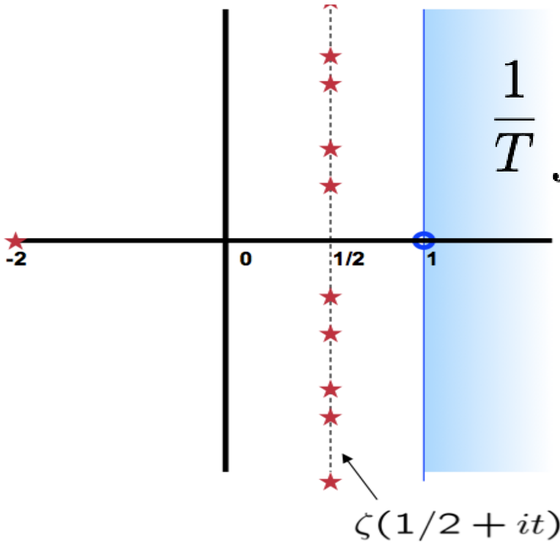
and

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Moments of the Riemann Zeta Function

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 dt \sim \log T$$

(Hardy and Littlewood, 1918)



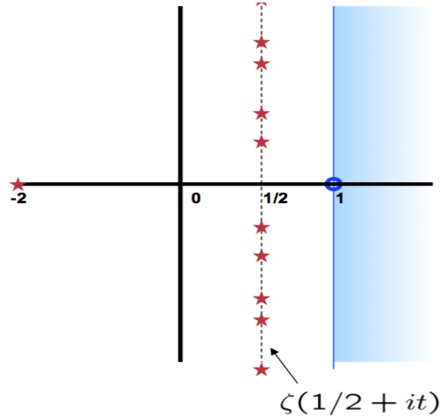
$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 dt \sim \frac{1}{2\pi^2}\log^4 T$$

(Ingham, 1926)

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \mathrm{Re}\, s > 1$$
$$= \prod_p (1-1/p^s)^{-1}$$

“Every moment brings a treasure; of its own especial pleasure;
Though the moments quickly die; greet them gaily as they fly!”

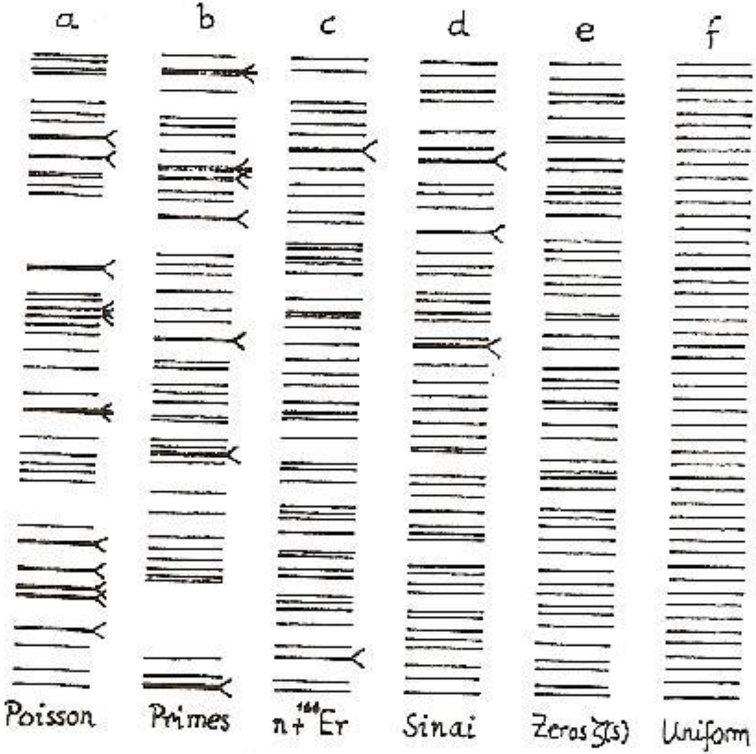
Density of zeros:



$$d(t) \sim \frac{1}{2\pi} \log \frac{t}{2\pi}$$

$$w_n = t_n \frac{1}{2\pi} \log \frac{t_n}{2\pi}, \quad t_n = \text{n}^{\text{th}} \text{ Riemann zero}$$

scale the Riemann zeros so that their average spacing is 1



From: M.L.Mehta,

Random Matrices, 3rd Ed,

Elsevier, 2004

Random Matrix Theory

- Developed in physics in 1950s
- Used to model excitation states of atomic nuclei
- Hamiltonian modelled by random matrix
- Spacing statistics of resonances predicted by eigenvalues of random matrices

Undergraduate degree in Physics, McMaster University, Hamilton, Canada


Hamilton Amateur Astronomers

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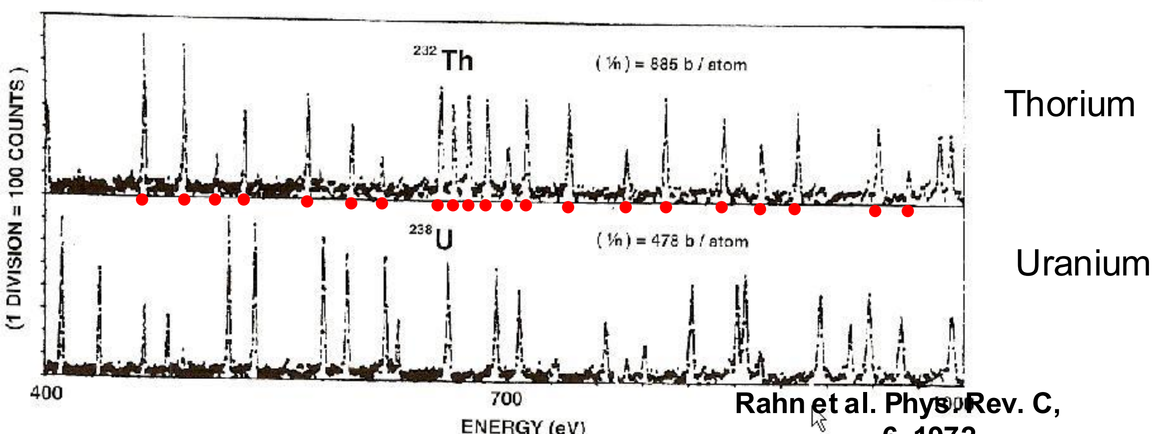
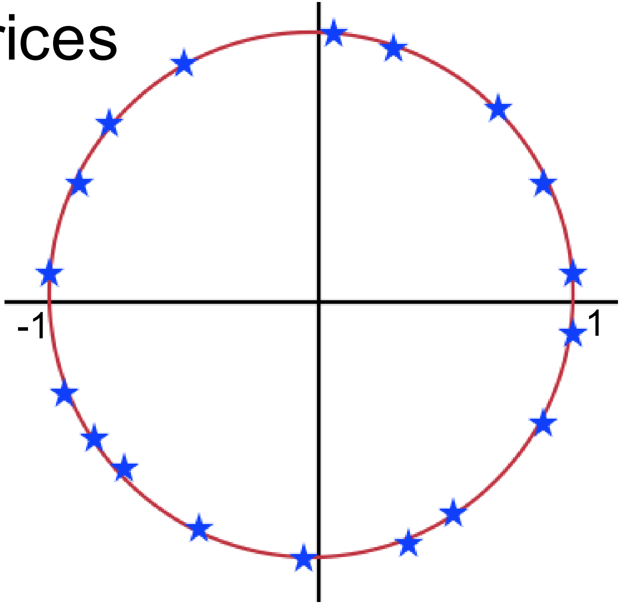


Rahn et al. Phys. Rev. C, 6, 1972

Random Unitary Matrices



A is an $N \times N$ unitary matrix:

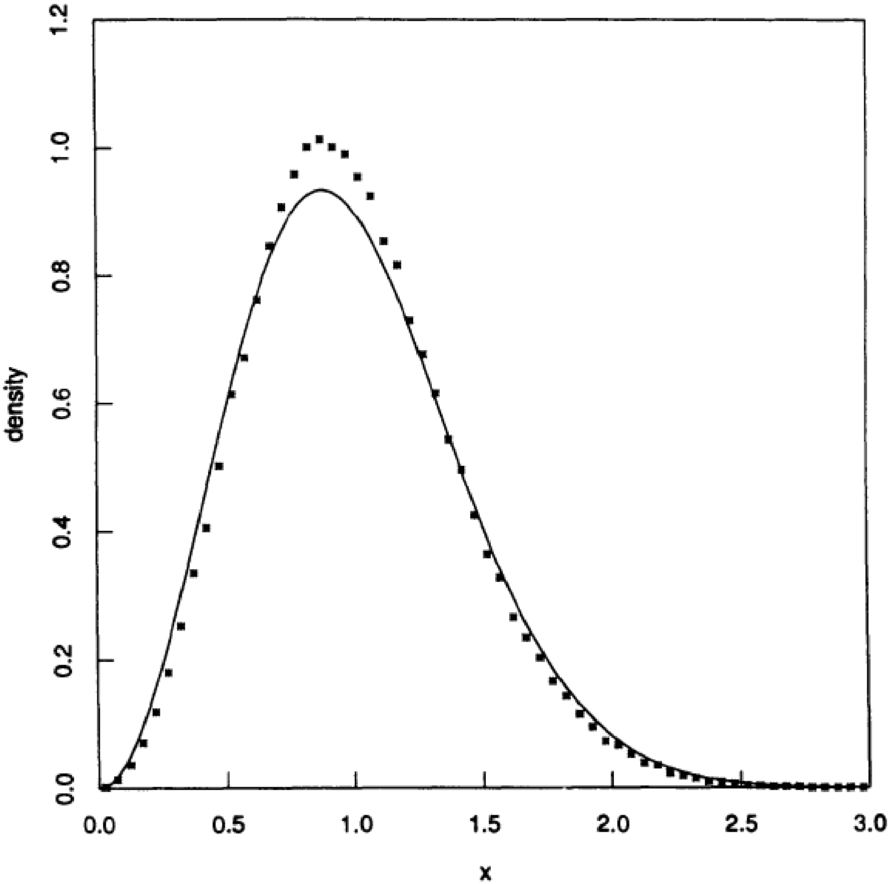
$AA^\dagger = A^\dagger A = I$

eigenvalues of A: $e^{i\theta_n}$

A is chosen randomly with respect to Haar measure on $U(N)$

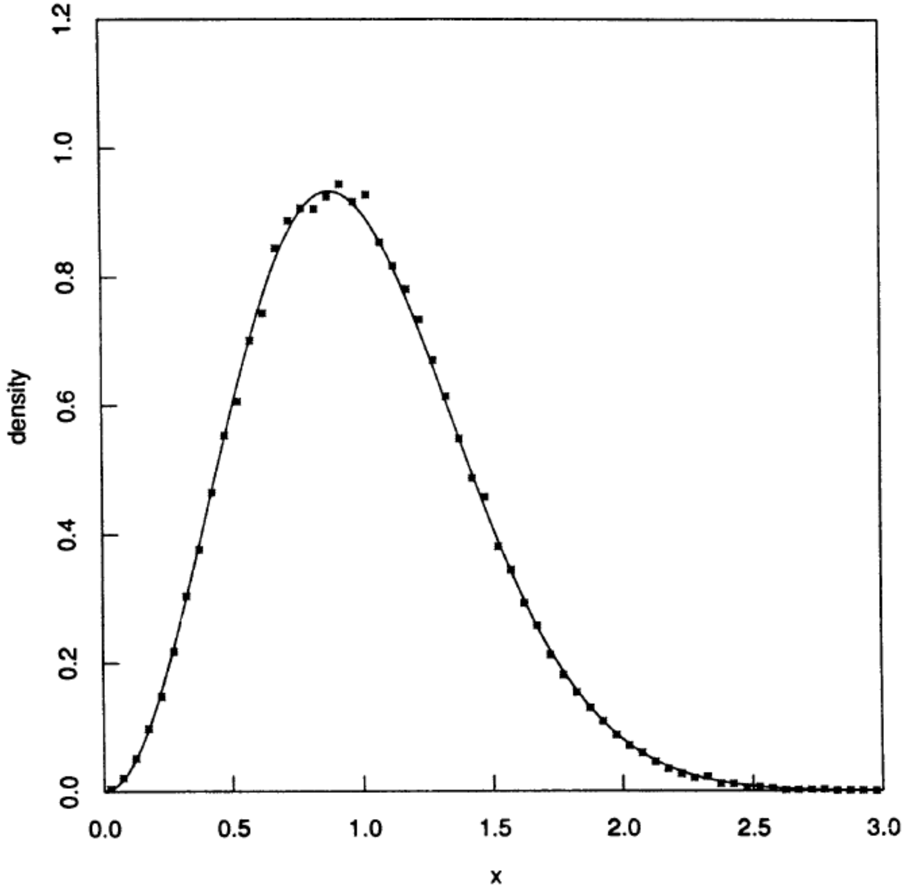
density of eigenphases: $\frac{N}{2\pi}$

Nearest Neighbor Spacing

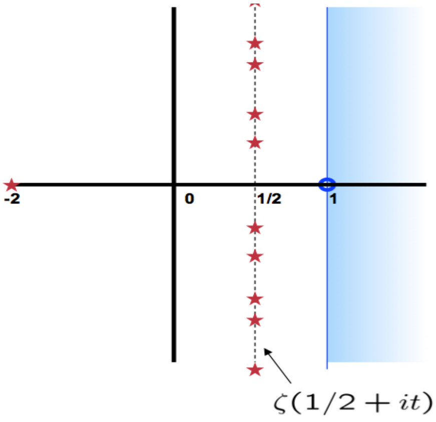


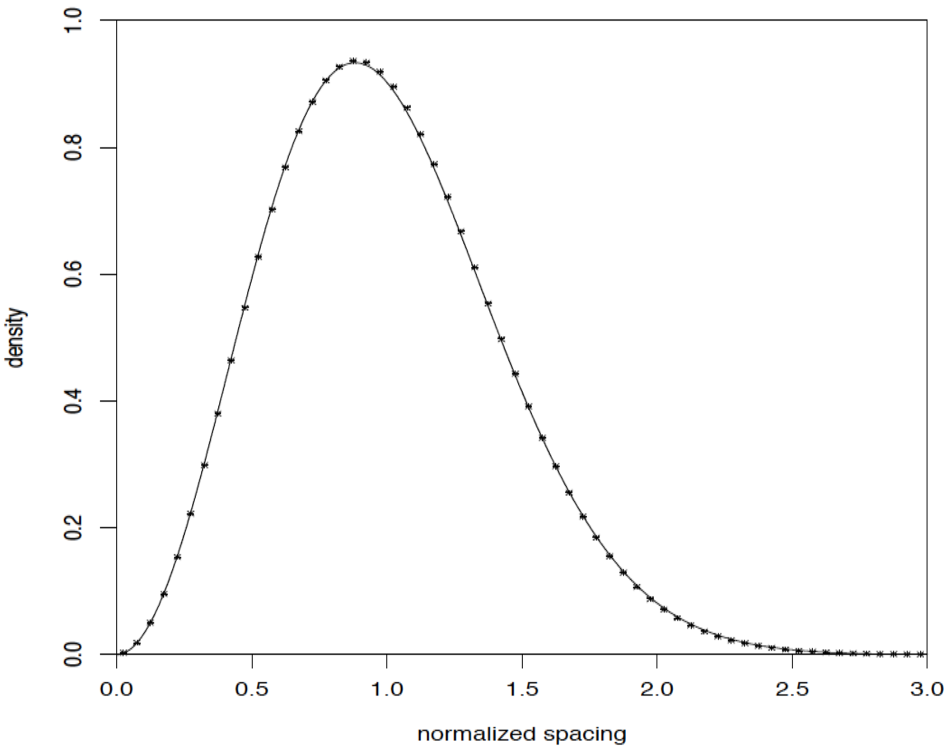
Probability density for distances between consecutive eigenvalues/ zeros

Using the first 100000 Riemann zeros – Picture by Andrew Odlyzko



10^5 zeros around the 10^{12}th zero





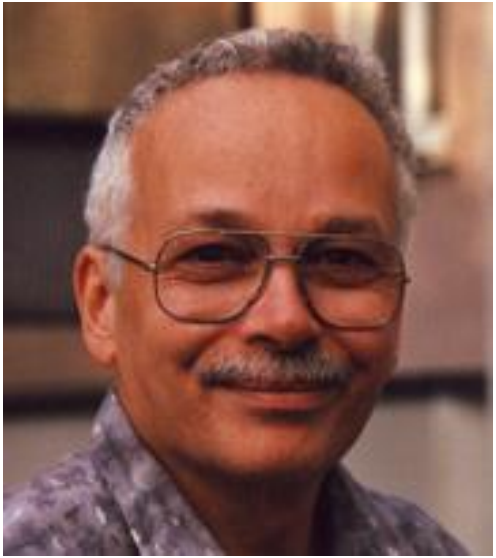
Picture by
A. Odlyzko

billion zeros
around the
$10^{16\text{ th}}$ zero

In 1972 these two met:

Freeman Dyson:
Physicist

Hugh Montgomery:
Mathematician



THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY 08540

SCHOOL OF NATURAL SCIENCES

April 7 1972

Dear Atle

The reference which Dr Montgomery wants is

M. L. Mehta, "Random Matrices"
Academic Press, N.Y. 1967.

Page 76 Equation 6.13
Page 113 Equation 9.61

showing that the pair-correlation function of zeros of the ζ-function is identical ~~which~~ with that of eigenvalues of a random complex (Hermitian or unitary) matrix of large order.

Freeman Dyson.

Moments of the Riemann Zeta Function

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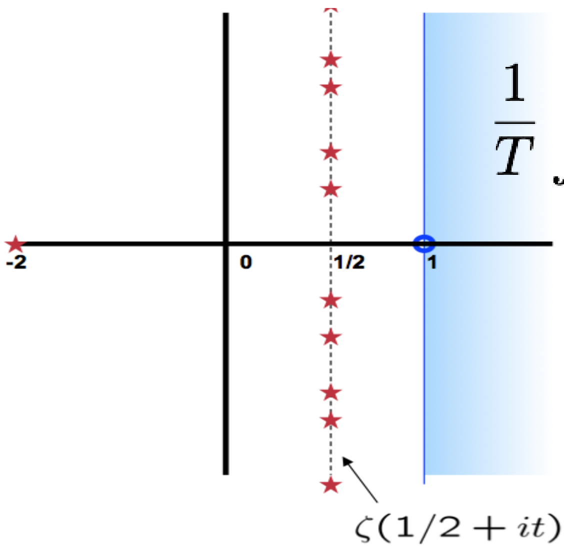

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$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \mathrm{Re}\, s > 1$$
$$= \prod_p (1-1/p^s)^{-1}$$

Conrey and Ghosh suggest that as $T \to \infty$:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda} dt \sim a_\lambda \frac{g_\lambda}{\Gamma(1+\lambda^2)} \log^{\lambda^2} T$$

with $a_\lambda = \prod_p \left[(1-1/p)^{\lambda^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+\lambda)}{\Gamma(\lambda) m!}\right)^2 p^{-m}\right]$,
a product over prime numbers

$$\frac{g_0}{\Gamma(1)} = 1$$

$$\frac{g_1}{\Gamma(1+1)} = 1 \quad \text{theorem – Hardy\&Littlewood 1918}$$

$$\frac{g_2}{\Gamma(1+2^2)} = \frac{1}{12} \quad \text{theorem – Ingham 1926}$$

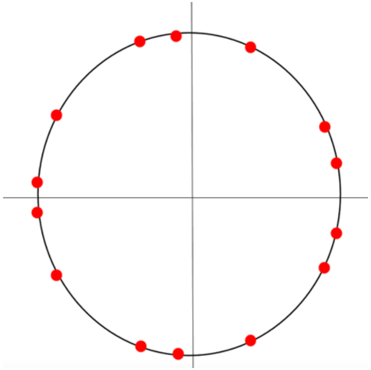
$$\frac{g_3}{\Gamma(1+3^2)} = \frac{42}{9!} \quad \text{conjecture – Conrey\&Ghosh 1992}$$

In 1996 Peter Sarnak…

…challenged Jonathan Keating

PhD in the School of Mathematics, University of Bristol

Characteristic polynomial:



$$\Lambda_A(s) = \prod_{n=1}^{N} \left(1 - s e^{-i\theta_n}\right)$$
$$= \det\left(I - A^\dagger s\right)$$

Equate densities of zeros:

$$\underset{\text{RZF}}{\frac{1}{2\pi} \log \frac{T}{2\pi}} = \underset{\text{RMT}}{\frac{N}{2\pi}}$$

Moments of the Riemann Zeta Function: Conjecture

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda} dt \sim a_\lambda f_\lambda \log^{\lambda^2} T$$

Moments of characteristic polynomials: Theorem

$$\int_{U(N)} |\Lambda_A(1)|^{2\lambda} dA_{Haar} \sim f_{\lambda,U(N)} N^{\lambda^2}$$

$$N \sim \log T$$

Conjecture (Keating and Snaith, 2000): $f_\lambda = f_{\lambda,U(N)}$

$$\mathrm{Re}\,\lambda > -1/2$$

$$\int_{U(N)} |\Lambda_A(1)|^{2\lambda} dA_{Haar}$$

$$= \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{n=1}^{N} |1 - e^{-i\theta_n}|^{2\lambda} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \dots d\theta_N$$

$$\prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}| = \det \begin{pmatrix} 1 & e^{i\theta_1} & e^{2i\theta_1} & \cdots & e^{(N-1)i\theta_1} \\ 1 & e^{i\theta_2} & e^{2i\theta_2} & \cdots & e^{(N-1)i\theta_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{i\theta_N} & e^{2i\theta_N} & \cdots & e^{(N-1)i\theta_N} \end{pmatrix}$$

$$(1-e^{-i\theta_1})(1-e^{-i\theta_2})(1-e^{-i\theta_3}) = 1-e^{-i\theta_1}-e^{-i\theta_2}-e^{-i\theta_3}+e^{-i\theta_1}e^{-i\theta_2}+e^{-i\theta_1}e^{-i\theta_3}+e^{-i\theta_2}e^{-i\theta_3}-e^{-i\theta_1}e^{-i\theta_2}e^{-i\theta_3}$$

$$(1-x_1)(1-x_2)(1-x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

Erwin Schrödinger Institute in Vienna

Jon Keating

Brian Conrey

$$\frac{0!\,1!\,2!\,3!}{4!\,5!\,6!\,7!} \stackrel{?}{=} \frac{24024}{16!}$$

Random matrix theory, number theory and the movie Proof: