Degenerate & singular mixed dimensional diffusion systems: A journey from modelling, and numerical methods, to well-posedness

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Nonlinear, degenerate, and coupled (mixed-dimensionally) systems arise in various applications of societal relevance, such as biofilm growth, reactive transport in fractured porous media, and cellular biology. For such problems, the biomass or the main-substrate is restricted to a lower-dimensional manifold embedded in a domain in d-dimensional space, and the evolution of the biomass density/substrate concentration exhibits degenerate and singular diffusion behaviour (possibly even cross-diffusion). The other equations (for nutrients/reagents) are defined on the bulk, and are of linear advection-reaction-diffusion type.

In our analysis, we propose a backward Euler time-discretisation of the problem where the reactive terms coupling the equations are estimated semi-implicitly. Thus, the equations are dimensionally decoupled and can be solved sequentially. A bulk-surface finite element method is used to discretise the system in space. Then, an iterative linearisation algorithm is proposed to solve the fully-discrete nonlinear problem on the discretised interface. It is proven that the iterations converge unconditionally even for degenerate/singular cases and for curved interfaces, thus, showing the existence of a solution to the fully-discrete nonlinear problem. Then, for a flat interface, the convergence of the fully-discrete solutions to the time-discrete solution is shown, and the order of convergence with respect to mesh size is estimated. Properties such as well-posedness, boundedness, and positivity of the time-discrete solutions are proven, and the existence of the time-continuous solutions is shown by passing the time-step size to zero. Uniqueness is proven by a contraction argument, and blow-up of solutions is investigated for homogeneous Neumann conditions. Numerical results validate the theoretical predictions. They also reveal a rich source of pattern formations exemplified by biofilm and skin-patterning cases.