Higher Dimensional Quasicrystals from Hyperbolic Honeycombs

Quasicrystals in Fundamental Physics

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with Latham Boyle

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Outlines and Punchlines

I. **Introduction and Background**
II. **Refining AdS/qCFT in 2d/1d**
III. **Defining Boundary Quasicrystals**
IV. **Higher-Dimensional Quasicrystals: $\{3,5,3\}$ Example**
V. **Conclusion**

Three Punchlines

1. Half-step refinements and $\{p,q\} \leftrightarrow \{q,p\}$ duality
2. Define boundary quasicrystals and "holographic foliations"
3. Higher dimensional quasicrystals and Thurston's Conjecture

Introduction and Background

- Point groups < $O(d)$
  - Infinitely many 2d point groups.
- Add translation symmetry
  - **Crystallographic restriction**
  - Only 2,3,4,6-fold symmetry around a point in 2d and 3d
  - 17 Wallpaper Groups in 2d.

"p6m" tiling

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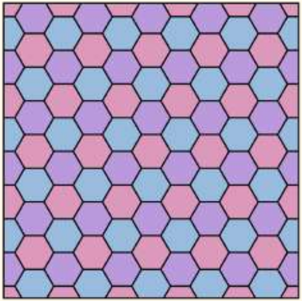
- **Regular tessellation** (“honeycomb”): uses only regular d-dimensional polytopes
- In the Euclidean plane, the angle of a regular p-gon $\{p\}$ is $(1-2/p)\pi$
  - Regular tilings of $\mathbb{R}^2$: {3,6}, {6,3}, {4,4}
  - In higher dimensions, must make sure that dihedral angles fit around edges, etc.

    E.g. Cubic honeycomb $\{4,3,4\}$ in 3d.



- We can tile other surfaces
  - Sphere (EdS) and hyperbolic space (EAdS) are other maximally symmetric spaces.

E.g. Tilings of sphere closely related to existence of 3d polyhedra

Truncated icosahedron

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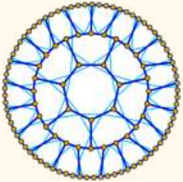
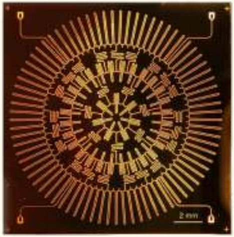
  - Sphere (EdS) and hyperbolic space (EAdS) are other maximally symmetric spaces.

E.g. Tilings of sphere closely related to existence of 3d polyhedra

- Hyperbolic space is more novel
  - Harder to intuit
  - Applications in art, biology, network theory, and physics
  - Hyperbolic band theory and quantum matter, tensor networks and error-correction, discrete holography





- In curved space, the angles of a regular p-gon gradually decrease to zero as the edges get bigger
  - Use this to tune p-gons to fit around a vertex

$$\frac{1}{p}+\frac{1}{q}\quad\begin{cases} > \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{S}^2 \\ = \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{E}^2 \\ < \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{H}^2 \end{cases}.$$

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$\mathbb{S}^2$

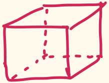
Tetrahedron

$\{3,3\}$

Octahedron

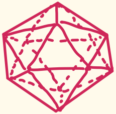
$\{3,4\}$

Cube



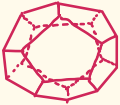
$\{4,3\}$

Icosahedron



$\{3,5\}$

Dodecahedron



$\{5,3\}$

- In curved space, the angles of a regular p-gon gradually decrease to zero as the edges get bigger
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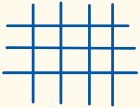
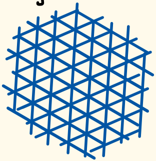
E^2

Square Lattice



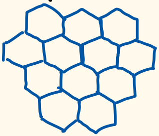
$\{4,4\}$

Triangular Lattice



$\{3,6\}$

Honeycomb Lattice



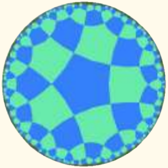
$\{6,3\}$

Regular Tilings in 2D

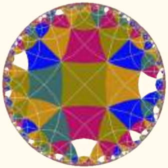
- In curved space, the angles of a regular p-gon gradually decrease to zero as the edges get bigger
  - Use this to tune p-gons to fit around a vertex

$$\frac{1}{p}+\frac{1}{q}\begin{cases} > \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{S}^2 \\ = \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{E}^2 \\ < \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{H}^2 \end{cases}.$$

$\mathbb{H}^2$ $\quad \ldots,$



$\{5,4\}$,



$\{4,5\}$, ...

- [Coxeter, ICM, 1954] classified **honeycombs in $\mathbb{H}^d$**

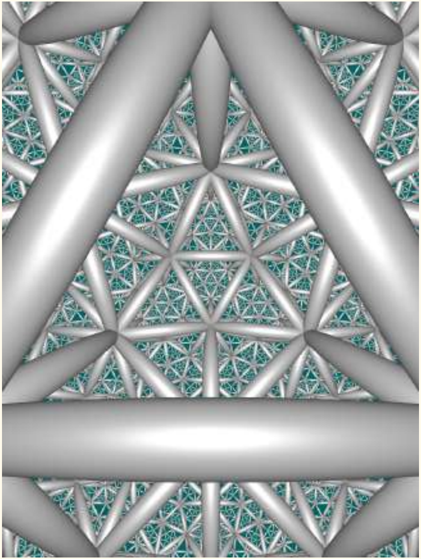
$\mathbb{H}^2$ $\{p, q\}$ where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$

$\mathbb{H}^3$ $\{4, 3, 5\}$, $\{5, 3, 4\}$, $\{5, 3, 5\}$, and $\{3, 5, 3\}$ [right]

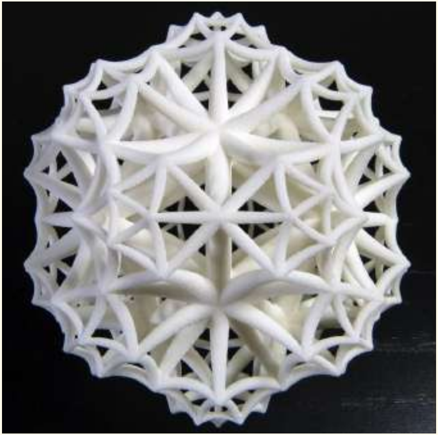
$\mathbb{H}^4$ $\{3, 3, 3, 5\}$, $\{5, 3, 3, 3\}$, $\{4, 3, 3, 5\}$, $\{5, 3, 3, 4\}$, and $\{5, 3, 3, 5\}$

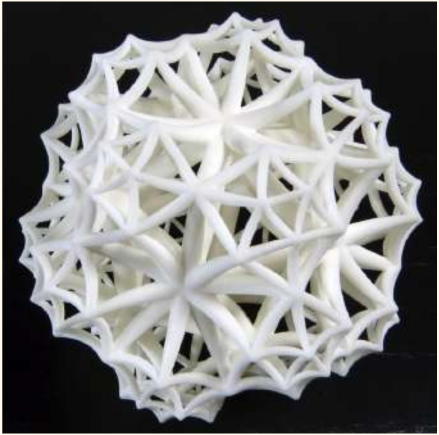
$\mathbb{H}^5$ Only some with non-finite cells and vertex figures.

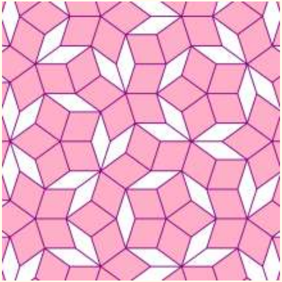
$\mathbb{H}^d$ When $d > 5$, none at all.



The {3,5,3} Hyperbolic Honeycomb Outside

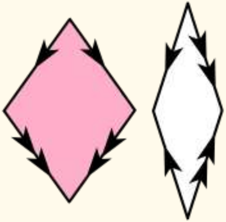




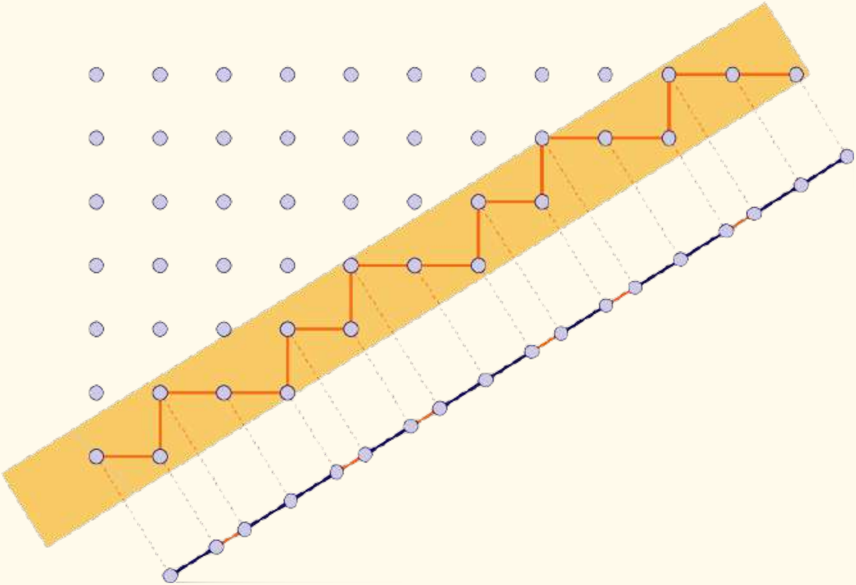


- **Quasicrystals** are aperiodic tilings which still possess some amount order.
  - **Aperiodic** means no translation symmetry.
  - Classically forbidden symmetries, e.g. 5-fold
  - Define by their diffraction pattern: reciprocal lattice is spanned by $\mathbb{Z}$-linear combinations of $> d$ basis vectors in d dimensions.
- There are many ways to create a quasicrystal (not all of which are equivalent).
  - Constructions highlight different properties.

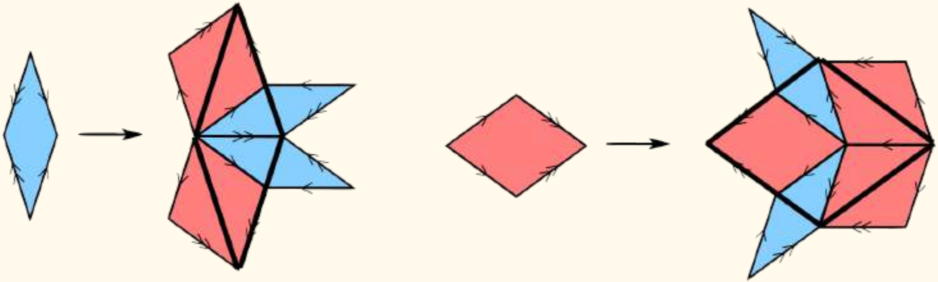
1. **Local Matching Rules**.
2. **Ammann Lines**. Long-range order
3. **Cut and Project**. Forbidden symmetries
4. **Substitution Rules**. Aperiodicity
5. **Pentagrid**. Defects



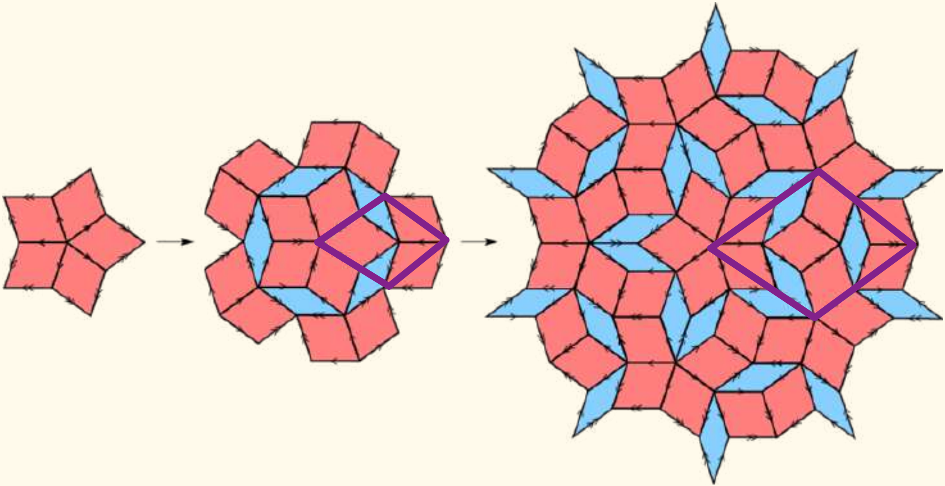
Cut and Project



Substitution Rules: Definition

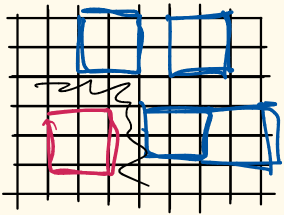


Substitution Rules: In Action



Substitution Rules: Implies Aperiodic

- Start with a finite set of (decorated) prototiles $\tau = \{T_1, \ldots, T_N\}$
  - ▶ Suppose there is a unique rule for grouping prototiles into larger "supertiles," with **scale factor** λ
  - ▶ Then τ is an aperiodic set.



Ex. Penrose tiling has two prototiles; supertiles have $\lambda = \frac{1}{2}(1+\sqrt{5})$.

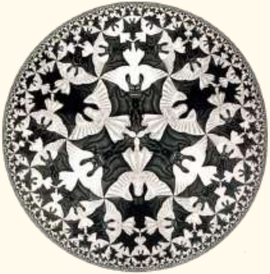
Ex. (Non-Example) There is no unique/local way to group an unmarked square tiling into larger squares. Domain walls can form.

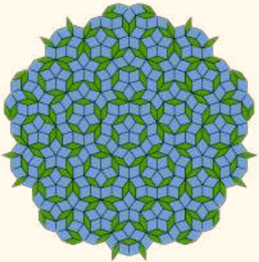
- Capture "combinatorial information" in the **substitution matrix**.
  - ! For aperiodic tilings, dominant eigenvalue is $\lambda^2 > 1$ and irrational.
  - ▶ Relative lengths of tiles, densities, etc. captured by eigenvectors

$$\begin{pmatrix} \text{Thin}' \\ \text{Thick}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \text{Thin} \\ \text{Thick} \end{pmatrix} \tag{1}$$

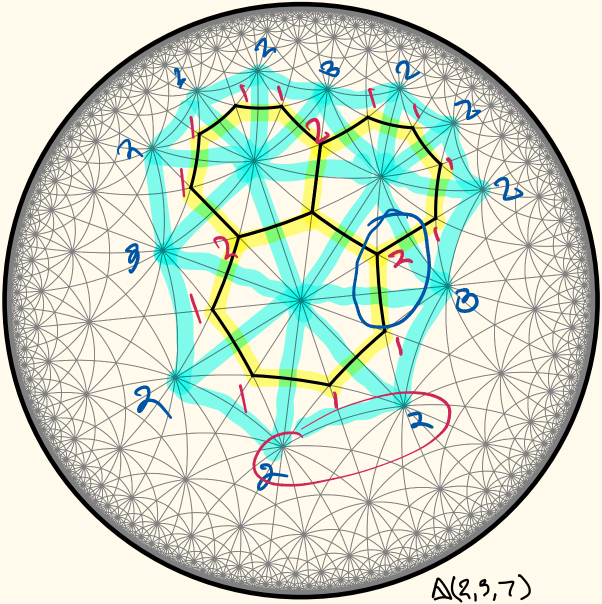
Summary of Background

- **Hyperbolic space** exists:
  - You can tile it with **regular tessellations**. There are infinitely many $\{p, q\}$ tilings in 2d, and a few more in higher dimensions.
- **Quasicrystals** exist:
  - Aperiodic. Classically forbidden symmetries. Long-range order.
  - Many constructions; most useful for us is existence of a unique (invertible) local **substitution rule** with irrational scale factor λ.
  - Most famous example is the Penrose tiling.





Refining AdS/qCFT in 2d/1d



$\Delta(2,3,7)$

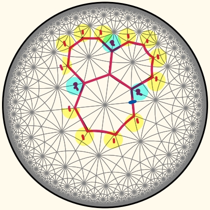
$= \{7,3\}$

$= \{3,7\}$

$1 \rightarrow 2^{1/2}\, 2^{1/2}$

$2 \rightarrow 2^{1/2}\, 3\, 2^{1/2}$

The Claim

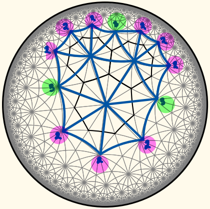


walk
rotate
walk

$\overset{\partial}{\longmapsto}$ $(1111\,2\,1111\,2\,1111\,2)$

Grow $\downarrow$

$\downarrow$ Inflate

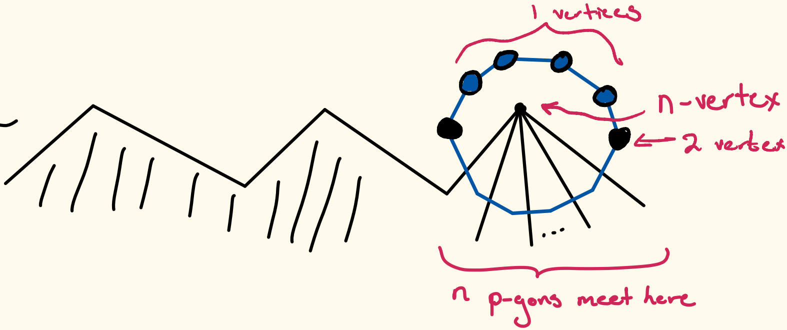


$\overset{\partial}{\longmapsto}$ $(222\,3\,222\,3\,222\,3)$

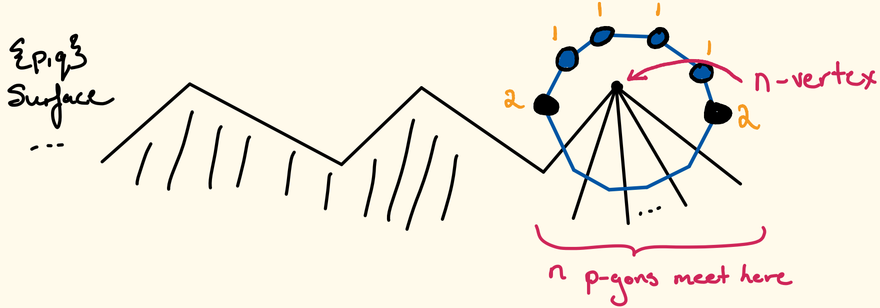
Take $N_{\text{inflation}} \to \infty$ to get full quasicrystal.

A General Rule in 2D/1D

$\{p,q\}$
Surface
...



Dual q-gon



- **General rule** can be written

$$n \mapsto \mathbf{2}^{1/2}\mathbf{1}^{q-n-2}\mathbf{2}^{1/2} \tag{2}$$

  - Gives sequence of **local maps**

$$\{p,q\} \mapsto \{q,p\} \mapsto \{p,q\} \mapsto \dots \tag{3}$$

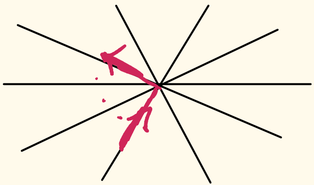
ADDITIONAL COMMENTS

- **Unique local rule** up to choice of MLD scheme
  - $\exists$ MLD schemes with no negative/fractional tiles, if desired
- **Identify tiles**/rules in terms of $\mathbf{1}$'s and $\mathbf{2}$'s via $\mathbf{3} = \mathbf{2}\mathbf{1}^{-1}\mathbf{2}$, $\mathbf{4} = \mathbf{2}\mathbf{1}^{-1}\mathbf{2}\mathbf{1}^{-1}\mathbf{2}$, ..., and generally (and uniquely):

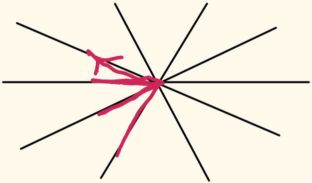
$$\boldsymbol{n} = \mathbf{2}(\mathbf{1}^{-1}\mathbf{2})^{n-2} \tag{4}$$

  - Negative tiles have clean local geometric interpretation
  - Can be used to reconstruct tiling from boundary

$3 \sim$



$21^{-1}2 \sim$



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  - Negative tiles have clean local geometric interpretation
  - Can be used to reconstruct tiling from boundary
- **Substitution matrix** written in terms of $\mathbf{1}$ and $\mathbf{2}$

$$M_{pq\mapsto qp} = \begin{pmatrix} q-3 & q-4 \\ 1 & 1 \end{pmatrix} \tag{5}$$

  - Product $M_{qp\mapsto pq}M_{pq\mapsto qp}$ is "usual full-step inflation matrix"
  - Can add all tiles $\mathbf{1}, \ldots, \boldsymbol{q}$, but only two non-zero eigenvalues means only two effective tiles after inflations

Defining Quasicrystals & Holographic Foliations

- Start with a **finite initial seed** of tiles S_0 and grow it layer by layer in the bulk $\rightsquigarrow$ boundary inflation

$$S_0 \mapsto S_1 \mapsto S_2 \mapsto \ldots . \tag{6}$$

  - For finite S_k: can turn simply connected to non-simply connected, or vice-versa; can turn disconnected into connected, etc.
  - After large (but finite) number of inflations N, will turn into one simply connected patch S_N, populated by only 2 species.
  - **Erases information** about the initial configuration, S_0

Defining QCs: Inflation from a Seed

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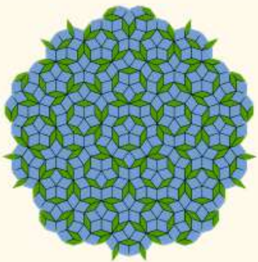
  - After large (but finite) number of inflations N, will turn into one simply connected patch S_N, populated by only 2 species.
  - **Erases information** about the initial configuration, S_0
- "Quasicrystal" lives on "boundary."
  - Technically not aperiodic or at boundary.
- In **infinite limit**, obtain a genuine aperiodic tiling at boundary

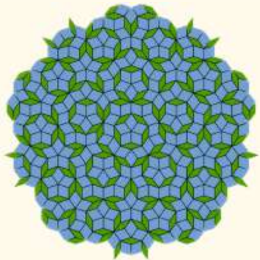
$$\text{“ 1d Quasicrystal} = \lim_{k\to\infty} S_k \text{ ”} \tag{7}$$

  - Not obviously well-defined/stable either

DEFINING QCS: DE BRUIN'S DEFLATIONS 1/2

- All complaints apply to Penrose tiling as well [de Bruijn, 81]
  - Inflation is not the best way to define the Penrose tiling.
  - Inflation gives (incorrect) impression that there is a small number of different Penrose tilings (based on initial seed)
  - There are an **uncountable infinity** of **locally indistinguishable** but globally distinct Penrose tilings





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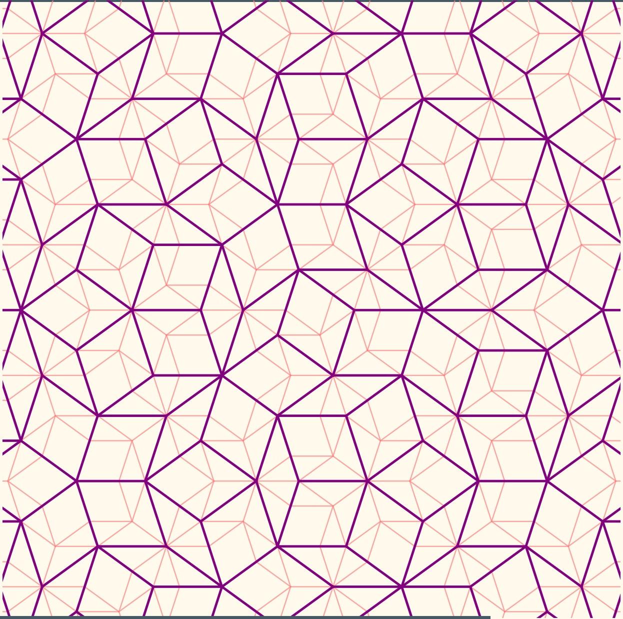
  - There are an **uncountable infinity** of **locally indistinguishable** but globally distinct Penrose tilings
- Class of all Penrose tilings is better defined using **deflation**
  - Given an infinite tiling of the plane by (unmarked) Penrose tiles, we ask: *is there a unique way to group the tiles into larger supertiles*? This is deflation.
  - If this can be done infinitely many times, all the objects in the sequence are valid Penrose tilings.
  - Inflation is important, but the key is having a unique local invertible rule. There are theorems that can guarantee the existence of this inverse.

Defining QCs: de Bruin's Deflations 2/2



Defining QCs: Holographic Foliations 1/2

- Our situation is identical
  - Do not start with a seed and inflate
  - There is an (uncountably infinite) class of 1d $\{p,q\}/\{q,p\}$ quasicrystals: take any (infinite) 1d surface in the $\{p,q\}/\{q,p\}$ triangulation; if it can be *uniquely* and *indefinitely* deflated, then **all of the surfaces *define* valid quasicrystals**.

$$\cdots \rightarrow S_{i-1} \rightarrow S_i \rightarrow S_{i+1} \rightarrow \cdots$$

$$\cap$$

$$\{\text{All } \{p,q\}|\{q,p\} \text{ QCs}\}$$

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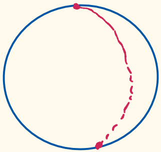
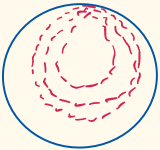
- This gives a new picture of the space of $\{p,q\}/\{q,p\}$ quasicrystals and where they live.
  - 1d quasicrystals **do not necessarily live on the boundary of $\mathbb{H}^2$**
  - They live on any infinite 1d surface which can be deflated
  - This includes slices through the bulk itself. Speculation: Karch-Randall or Randall-Sundrum braneworld scenarios?





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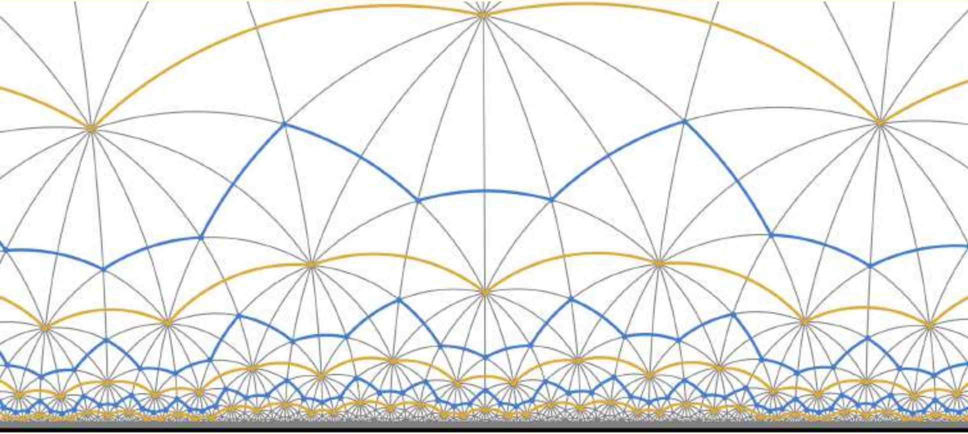
  - This includes slices through the bulk itself. Speculation: Karch-Randall or Randall-Sundrum braneworld scenarios?
- Bulk-boundary picture gives an interpretation where all 1d quasicrystal inflations and deflations in a sequence live in the same 2d hyperbolic spacetime.
  - Defines a **holographic foliation** of $\mathbb{H}^2$.

Defining QCs: Holographic Foliations 2/2

- Concrete example of a sequence and holographic foliation comes from slicing by (approximate) horospheres. i.e. constant z height in the upper-half plane



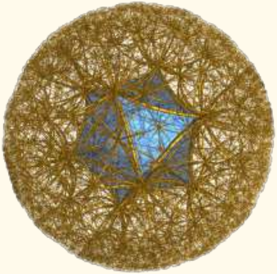
Higher-D Quasicrystals

Higher Dimensions and a Conjecture of Thurston

- A conceptual problem/test:
  - 1d is too topological: no curvature
  - Is this really a relationship between geometry of hyperbolic tessellations and quasicrystals?
- Can we identify a similar relationship in higher dimensions?
  - Recall in $\mathbb{H}^3$ there were 4 regular tessellations: $\{4,3,5\}$, $\{5,3,4\}$, $\{5,3,5\}$, and $\{3,5,3\}$ [right]
  - Extend growth rules to higher dimensions

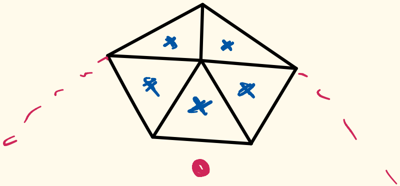
Conjecture: BDF (2019) Thurston (?)

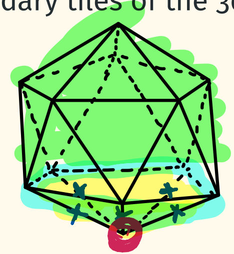
The boundary of the {3,5,3} lattice looks like the Penrose tiling.



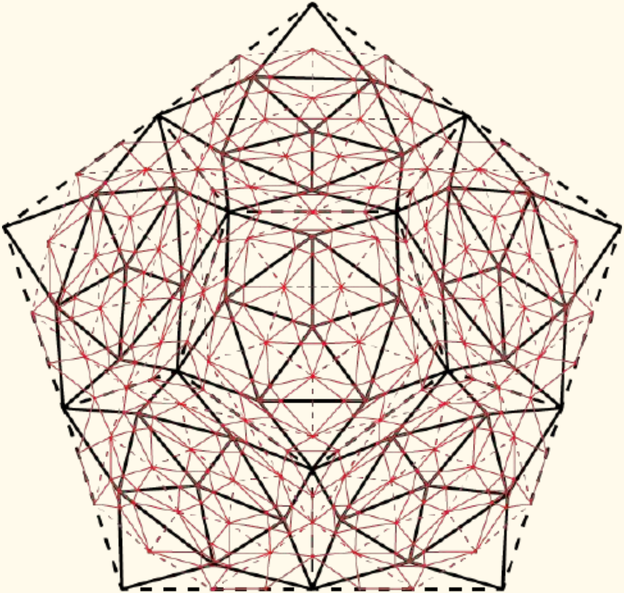
Growing {3,5,3}

Think carefully about how the bulk 3d patch grows. Making a careful analogy to the 2d/1d examples, we will see how 2d boundary tile data is encoded in boundary tiles of the 3d patch.

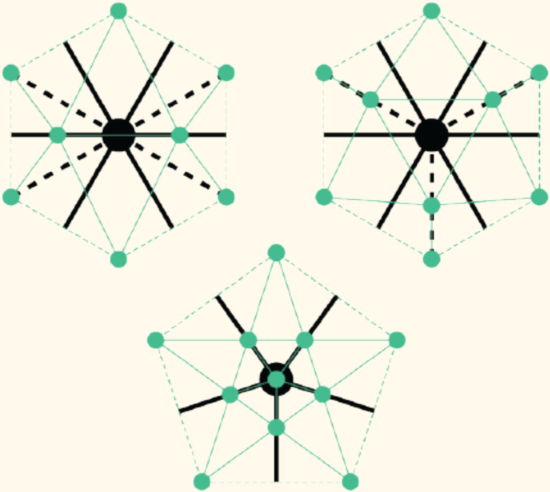




{3,5,3} Projected



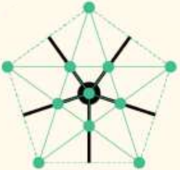
The Tiles

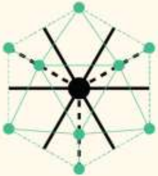


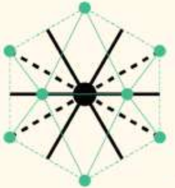
3d Substition Matrices

- 3 natural tiles in the growth of a single icosahedral cell: 1-vertex V_1, 3-vertex V_3, and 4-vertex V_4 tiles.
  - More complicated seeds give more complicated tiles; asymptotes to three after many inflations.
- Defines a **consistent 2d aperiodic substitution rule**

$$\begin{pmatrix} \bar{V}_1 \\ \bar{V}_3 \\ \bar{V}_4 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 2 \\ \frac{5}{3} & 0 & -\frac{2}{3} \\ 0 & \frac{3}{2} & 2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_3 \\ V_4 \end{pmatrix} \tag{8}$$

$V_1 \sim$ 

$V_3 \sim$ 

$V_4 \sim$ 

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  - Has (right) eigenvalues and eigenvectors

$$\lambda_\pm = \varphi_\pm^4 \,, \qquad v_\pm^T = \left(\varphi_\pm^4, \varphi_\pm, \tfrac{1}{2}\right) \,, \tag{9}$$

$$\lambda_0 = 1 \,, \qquad v_0^T = \tfrac{1}{6}\,(0, -2, 3) \,. \tag{10}$$

  - As always, components of dominant eigenvector v_+ give asymptotic relative frequencies
  - v_0 gives **Gaussian curvature** of tiles. Ensures Euler characteristic χ is preserved under inflation.

Boyle-Dickens-Flicker-Thurston Conjecture

Claim: *the boundary of the $\{3,5,3\}$ tiling is the Penrose tiling.*

1. Our 5-fold symmetric tiling has 3, not 2, tiles.
   - **Apply χ as constraint**; view tiling as built from $\boldsymbol{V_3}$ and $\boldsymbol{V_4}$ only.
   - The substitution matrix becomes

$$M'_{353} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = M^2_{\text{Penrose}} \tag{11}$$

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3. Each of the uncountably many PTs has (countably) infinite neighbourhoods of arbitrarily large 5-fold symmetry.
   - Fact: Only 4 PTs have **global** 5-fold rotational symmetry. Under inflation, they cycle into each other with period 4:

$$\mathrm{Sym}\,A \mapsto \mathrm{Sym}\,B \mapsto (\mathrm{Sym}\,A)^R \mapsto (\mathrm{Sym}\,B)^R \mapsto \dots \tag{12}$$

   - Our substitution rules map a globally 5-fold symmetric tiling to itself under one inflation, but the eigenvalues suggest it needs 2 inflations. So it must not be MLD to PTs.

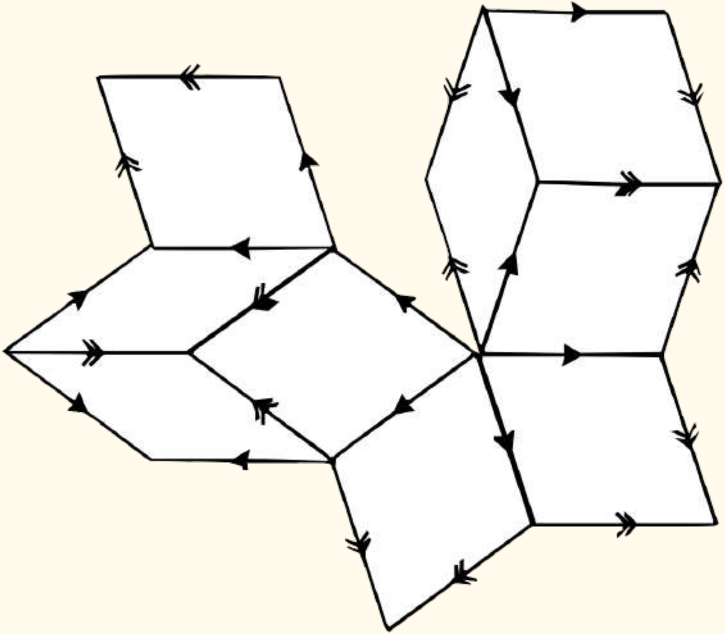
Conclusion

Outlines and Punchlines

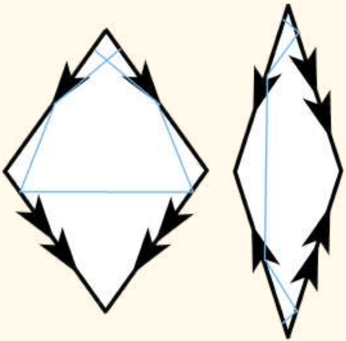
1. Half-step refinements and $\{p,q\} \leftrightarrow \{q,p\}$ duality
   - Refined and related growth rules of $\{p,q\}$ and $\{q,p\}$ tilings
   - New useful algebraic rules for manipulating the tilings
2. Define boundary quasicrystals and “holographic foliations”
   - Clarified definition of boundary quasicrystal
   - Introduced **holographic foliations**
3. Higher dimensional quasicrystals and Thurston’s Conjecture
   - Sketched the growth of the $\{3,5,3\}$-tiling
   - Can give geometric interpretation to the inflation matrix and associated data
   - Some **new 5-fold symmetric aperiodic tessellation** appears on the boundary (not the Penrose tiling)

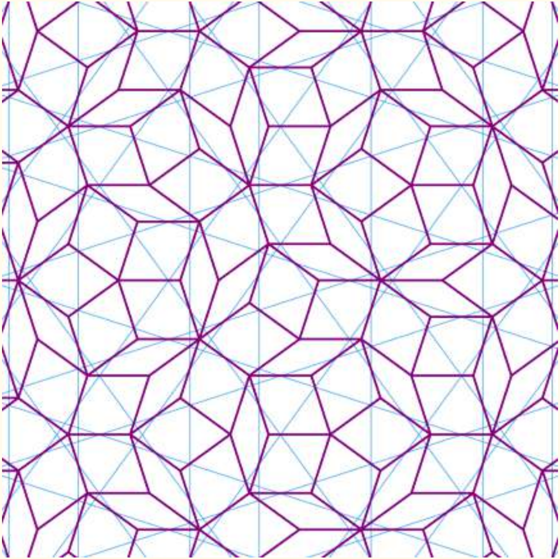
Fin

Failed Tiling



Ammann Lines





No 5-Fold Symmetry in 2D/3D

1. Pick a site of 5-fold symmetry
2. Take your favourite nearby lattice site and hit it with your 5-fold symmetry action.
3. Generates a regular pentagon of sites in a plane of the lattice.
4. All pentagon displacement vectors must be in the lattice.
5. Add them up at a single point to generate sites forming a smaller pentagon.