Formal languages and physical systems

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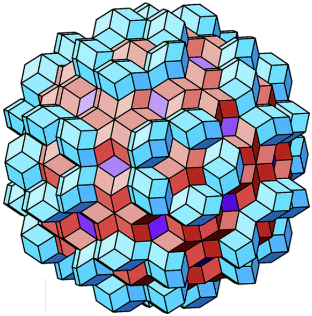
Quasicrystals in Fundamental Physics
International Centre for Mathematical Sciences
Edinburgh, July 2026

This talk is based on:

- Fernandes, Francesca; Marcolli, Matilde, *Formal languages, spin systems, and quasicrystals*. J. Geom. Phys. 216 (2025), Paper No. 105568, 28 pp.

Motivating example: icosahedral quasicrystal

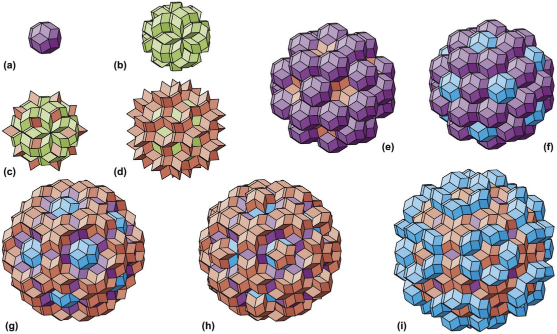


icosahedral packing with rhombic triacontahedron, rhombic dodecahedron, rhombic icosahedron, and prolate rhombohedra prototiles

different (equivalent) constructions

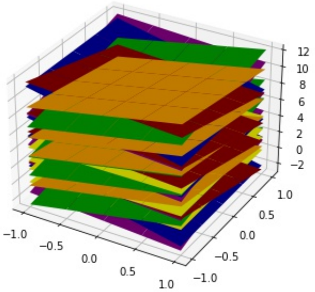
- Ammann rhombohedral tiling
- Danzer's *ABCK* tiling
- Socolar-Steinhardt tiling

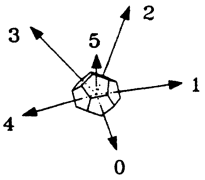
- Packings for quasicrystals are comprised of polyhedra prototiles, and can be constructed using inflation and deflation rules
- replace every prototile in the packing with other prototiles arranged in a specific orientation



- complicated rules on decomposition of prototiles in inflation/deflation
- Alternative description: dual Amman quasilattice (Socolar-Steinhardt)

dual Amman quasilattice





$$\vec{e}_j = \left(\frac{2}{\sqrt{5}}\cos\left(\frac{2\pi j}{5}\right), \frac{2}{\sqrt{5}}\sin\left(\frac{2\pi j}{5}\right), \frac{1}{\sqrt{5}}\right) \text{ for } j = 1, \ldots, 4, \qquad \vec{e}_5 = (0, 0, 1)$$

- Ammann quasilattice for the icosahedral quasicrystal
- each set of colored planes orthogonal to an icosahedral star vector $\vec{e}_j$
- planes in each set alternate $\mathbb{L}$ (long) and $\mathbb{S}$ (short) spacing
- Fibonacci hexagrid

- inflation/deflation: each system of planes according to rules for a 1D quasicrystal (Fibonacci)
- Amman quasilattice: Fibonacci hexagrid that is locally congruent to its deflation
- Fibonacci $D0L$ system $\mathcal{D} = (\Sigma, \phi, \Omega)$ with $\mathfrak{A} = \{\mathbb{S}, \mathbb{L}\}$, $\Omega = \mathbb{L}$, and $\phi : \Sigma^* \to \Sigma^*$

$$\phi(\mathbb{L}) = \mathbb{L}\,\mathbb{S}\,, \qquad \phi(\mathbb{S}) = \mathbb{L}$$

$$\mathcal{L}_{\mathcal{D}} = \{\Phi^n(\mathbb{L})\}_{n\in\mathbb{N}}$$

- Note: ϕ applies everywhere at once
- Fibonacci 1D quasicrystal equivalent to

$$\phi(\mathbb{S}) = \frac{\mathbb{L}}{2}\frac{\mathbb{L}}{2}, \qquad \phi(\mathbb{L}) = \frac{\mathbb{L}}{2}\mathbb{S}\frac{\mathbb{L}}{2}.$$

- word $w = \phi(u)$ in $\mathcal{L}_{\mathcal{D}}$ for some other $u \in \mathcal{L}_{\mathcal{D}}$ gives 1D chain S_w where each spacing $\mathbb{S}$ in chain S_u replaced by pattern of spacings $\frac{\mathbb{L}}{2}\frac{\mathbb{L}}{2}$ and each $\mathbb{L}$ spacing by pattern $\frac{\mathbb{L}}{2}\mathbb{S}\frac{\mathbb{L}}{2}$.
- replacing S_u with S_w an inflation of the 1D quasilattice

Formal Languages

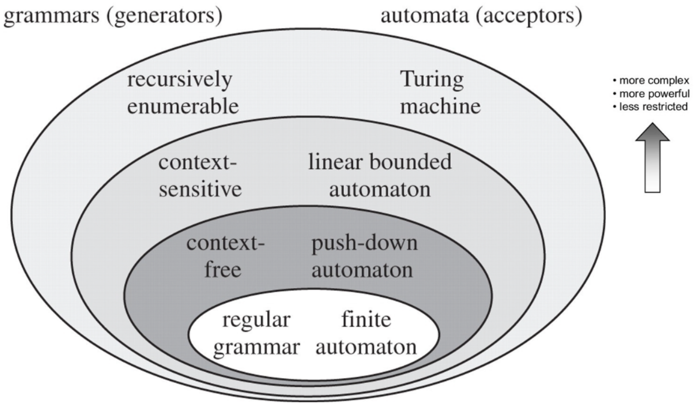
- $\mathfrak{A}$ be finite alphabet
- $\mathfrak{A}^*$ set of all strings of letters in $\mathfrak{A}$ of arbitrary (finite) length, free monoid on $\mathfrak{A}$ with concatenation product (unit empty word ϵ)
- *phrase structure grammar* $\mathcal{G} = (\mathfrak{Q}, \mathfrak{A}, P, S)$ with finite set $\mathfrak{Q}$ of "non-terminal symbols", alphabet $\mathfrak{A}$ as "terminal symbols", S start symbol in $\mathfrak{Q}$, finite set P of *production rules*
  - grammar is *regular* if all production rules $A \to a$ or $A \to Aa$ (left-regular) for some $A \in \mathfrak{Q}$ and some $a \in \mathfrak{A}$ (or $A \to aA$ right-regular)
  - grammar is *context-free* if the production rules $A \to \alpha$ with α string of terminals and non-terminals
  - other more general classes (context-sensitive, unrestricted), Chomsky hierarchy
- formal language $\mathcal{L} = \mathcal{L}_{\mathcal{G}}$ computed by grammar $\mathcal{G}$: subset $\mathcal{L} \subset \mathfrak{A}^*$ of words in the terminal symbols obtained from a composition of production rules of $\mathcal{G}$ starting with S

Chomsky hierarchy theorem: machine recognition

- regular languages can be recognized (computed) by a finite state automaton
- context-free languages by a pushdown stack automaton
- context-sensitive languages by a linear bounded automaton
- unrestricted by a Turing machine

We will consider only regular, context-free, and a class of mildly context-sensitive languages (multiple context-free)

Chomsky hierarchy



finite state automata (FSA)

- $\mathcal{M} = (\mathfrak{Q}, F, \mathfrak{A}, \tau, q_0)$ with finite set $\mathfrak{Q}$ of states, subset $F \subset \mathfrak{Q}$ of final states, initial state q_0, and set of transitions

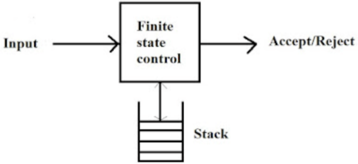
$$\tau \subset \mathfrak{Q} \times \mathfrak{A} \times \mathfrak{Q}$$

pushdown stack automata (PSA)

- $\mathcal{M} = (\mathfrak{Q}, F, \mathfrak{A}, \Gamma, \tau, q_0, z_0)$ with $\mathfrak{Q}, F, \mathfrak{A}, q_0$ as in FSA and with Γ stack alphabet and set of transitions

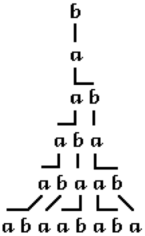
$$\tau \subset \mathfrak{Q} \times (\mathfrak{A} \cup \{\epsilon\}) \times \Gamma \times \mathfrak{Q} \times \Gamma^*$$

with an initial symbol $z_0 \in \Gamma$ for the stack (computation can end by final state or by empty stack)



D0L languages

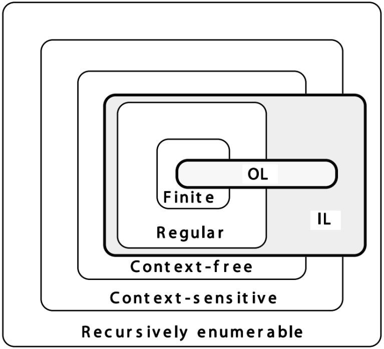
- Aristid Lindenmayer's L-systems (0L = context-free, IL = context-sensitive) rewriting systems for modeling branching processes in biology
- different from context-free languages: production rules apply everywhere in parallel, not sequentially



example of D0L derivation

- D0L-system: $\mathcal{D} = (\Sigma, \Phi, \Omega)$ with alphabet Σ and $\Phi : \Sigma^* \to \Sigma^*$ monoid homomorphism, initial word Ω, language $\mathcal{L}_{\mathcal{D}} = \{\Phi^n(\Omega), \, n \in \mathbb{N}\}$

- some context-sensitive languages can be obtained by $0L$-systems



0L and IL systems and the Chomsky hierarchy

D0L system and CF language

- Fibonacci CF language $\mathcal{L}_{\rm Fib}$: grammar $\mathcal{G} = (\mathfrak{Q}, \mathfrak{A}, P, S)$ with nonterminals $\mathfrak{Q} = \{S, A, B\}$, terminals $\mathfrak{A} = \{\mathbb{S}, \mathbb{L}\}$, and production rules

$$S \to A\,, \quad A \to AB\,, \quad B \to A\,, \quad A \to \mathbb{L}\,, \quad B \to \mathbb{S}$$

- difference with D0L: production rules applied sequentially (locally) rather than simultaneously (globally)
- D0L system as symmetries (endomorphisms) of Fibonacci CF language

$$\Phi : \mathcal{L}_{\rm Fib} \to \mathcal{L}_{\rm Fib}$$

$$\mathcal{X}_\Phi(\mathcal{L}_{\rm Fib}) = \mathcal{L}_{\rm Fib}$$

General idea

- words of CF language $\mathcal{L}_{\rm Fib}$ represent finite 1D spin chains, seeds of growth of 1D quasicrystal structure
- action of D0L system $\Phi : \mathcal{L}_{\rm Fib} \to \mathcal{L}_{\rm Fib}$ represent inflation rules and growth from initial word $\Omega \in \mathcal{L}_{\rm Fib}$ of a Fibonacci 1D quasicrystal
- a general framework for formal languages and spin chains: formal languages as objects of a category, with transductions (sequences to sequences transformations) as morphisms
- mapping functorially to a category with objects spin chains and morphisms that are decimations and renormalization maps

Rational correspondence (rational relation) $X = (\mathfrak{A}, \alpha, \beta, \mathcal{L})$

- between two monoids M, M'
- subset $X \subset M \times M'$ such that
- exists an alphabet $\mathfrak{A}$ and two morphisms of monoids $\alpha : \mathfrak{A}^* \to M$ and $\beta : \mathfrak{A}^* \to M'$
- exists a regular language $\mathcal{L} \subset \mathfrak{A}^*$ with

$$X = \{(\alpha(w), \beta(w)) \in M \times M' \,|\, w \in \mathcal{L}\}$$

Note: *correspondences* in general replace functions (equivalently defined by their graph) with more general subsets of the product (like multivalued functions): composition?

Composition

- monoids M, M', M'' and rational correspondences $X \subset M \times M'$ and $Y \subset M' \times M''$
- $X = (\mathfrak{A}, \alpha, \beta, \mathcal{L})$ and $Y = (\mathfrak{B}, \alpha', \beta', \mathcal{L}')$ (assume $\mathfrak{A} \cap \mathfrak{B} = \emptyset$)
- composition

$$Y \circ X = ((\mathfrak{A} \cup \mathfrak{B})^*, \alpha \circ \pi_{\mathfrak{A}}, \beta' \circ \pi_{\mathfrak{B}}, \hat{\mathcal{L}})$$

with $\hat{\mathcal{L}} \subset \pi_{\mathfrak{A}}^{-1}(\mathcal{L}) \cap \pi_{\mathfrak{B}}^{-1}(\mathcal{L}')$ given by

$$\hat{\mathcal{L}} = \{w \in \pi_{\mathfrak{A}}^{-1}(\mathcal{L}) \cap \pi_{\mathfrak{B}}^{-1}(\mathcal{L}') \,|\, \beta \circ \pi_{\mathfrak{A}}(w) = \alpha' \circ \pi_{\mathfrak{B}}(w)\}$$

with $\pi_{\mathfrak{A}} : (\mathfrak{A} \cup \mathfrak{B})^* \to \mathfrak{A}^*$ and $\pi_{\mathfrak{B}} : (\mathfrak{A} \cup \mathfrak{B})^* \to \mathfrak{B}^*$ projections

- for rational correspondences problem: composition need not preserve rationality ($\hat{\mathcal{L}}$ need not be regular), but OK for a subclass: rational transductions

Rational transductions

- rational correspondence $X = (\mathfrak{A}, \alpha, \beta, \mathcal{L}_{reg})$
- monoids M, M' are free monoids $\mathfrak{B}^*, \mathfrak{C}^*$ with alphabets $\mathfrak{B}, \mathfrak{C}$
- $\mathcal{L}_{reg}$ is a regular language

view a rational transduction as a map from languages inside $\alpha(\mathfrak{A}^*)$ to languages inside $\mathfrak{C}^*$

- maps $\Omega \subset \alpha(\mathfrak{A}^*) \subset \mathfrak{B}^*$ to $X(\Omega) \subset \mathfrak{C}^*$ by

$$X(\Omega) = \beta(\alpha^{-1}(\Omega) \cap \mathcal{L}_{reg})$$

- if Ω regular then $X(\Omega)$ regular
- if Ω context-free then $X(\Omega)$ context-free

- Note: resolves problem of rational correspondences because

$$\beta(\mathcal{L}_{reg}) \cap \alpha'(\mathcal{L}'_{reg}) \subset \mathfrak{C}^*$$

  is a regular language by closure of regular languages under homomorphisms *of free monoids* and under intersections

- Note: would not work if in X have $\mathcal{L}$ context-free instead of $\mathcal{L}_{reg}$ regular (intersection of context-free is not always context-free, but intersection or regulars is regular and intersection of regular and context-free is context-free

Transducers

- regular and context-free languages are computed by FSAs and PSAs
- rational transductions are computed by transducers
- $\mathcal{T} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{Q}, F, \tau, q_0)$ with:
- input alphabet $\mathfrak{A}$, an output alphabet $\mathfrak{B}$
- set of states $\mathfrak{Q}$ with initial state q_0 and set F of final states
- transitions
$$\tau \subset \mathfrak{Q} \times \mathfrak{A}^* \times \mathfrak{B}^* \times \mathfrak{Q}$$
- rational transduction obtained with $\mathcal{L}_{reg}$ the set of paths $\mathcal{P}(q_0, F) = \cup_{q\in F}\mathcal{P}(q_0, q)$, with $\mathcal{P}(q, q')$ set of paths $(q_1, a_1, b_1, q_2)\cdots(q_n, a_n, b_n, q_{n+1})$ with $q = q_1$, $q' = q_{n+1}$ and $(q_i, a_i, b_i, q_{i+1}) \in \tau$

Categorical viewpoint

- Formal languages (regular or context-free) with rational transduction as objects and morphisms of a category
- automata (FSA or PSA) and transducers as objects and morphisms of a category
- machine recognition of formal languages as a functor between these categories

Why category theory? Very convenient way of expressing compatibility (functoriality) of relations between different types of objects, and optimization (universal properties)

Category of context-free formal languages $\mathcal{CFL}$

- objects: context-free languages $\mathcal{L}$
- non-identity morphisms $X \in Mor_{\mathcal{CFL}}(\mathcal{L}, \mathcal{L}')$ of $\mathcal{L} \subset \mathfrak{B}^*$ and $\mathcal{L}' \subset \mathfrak{C}^*$ are rational transductions

$$X = (\mathfrak{A}, \alpha, \beta, \mathcal{L}_{reg}) \subset \mathfrak{B}^* \times \mathfrak{C}^*$$

- can take $\mathcal{L} \subset \alpha(\mathfrak{A}^*)$ and $\mathcal{L}' \subset \beta(\mathfrak{A}^*)$, or else intersect: free monoid is regular language, direct and inverse image of regular language is regular, so are $\alpha(\mathfrak{A}^*)$ and $\beta(\mathfrak{A}^*)$, intersection of CF and regular is CF so are $\mathcal{L} \cap \alpha(\mathfrak{A}^*)$ and $\mathcal{L}' \cap \beta(\mathfrak{A}^*)$
- need $X(\mathcal{L}) \subset \mathcal{L}'$ (or $X(\mathcal{L} \cap \alpha(\mathfrak{A}^*)) \subseteq \mathcal{L}' \cap \beta(\mathfrak{A}^*)$)

composition of morphisms is well defined

- $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ context free (over disjoint alphabets $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$), composition $\circ : Mor_{\mathcal{CFL}}(\mathcal{L}, \mathcal{L}') \times Mor_{\mathcal{CFL}}(\mathcal{L}', \mathcal{L}'')$
- $\beta(\mathcal{L}_{reg}) \cap \alpha'(\mathcal{L}'_{reg}) \subset \mathfrak{C}^*$ and $\pi_{\mathfrak{A}}^{-1}(\mathcal{L}_{reg}) \cap \pi_{\mathfrak{A}'}^{-1}(\mathcal{L}'_{reg}) \subset (\mathfrak{A} \cup \mathfrak{A}')^*$ are regular (homomorphisms of free monoids)
- also regular language:

$$\hat{\mathcal{L}} = (\beta\pi_{\mathfrak{A}})^{-1}(\beta(\mathcal{L}_{reg}) \cap \alpha'(\mathcal{L}'_{reg})) =$$

$$(\alpha'\pi_{\mathfrak{A}'})^{-1}(\beta(\mathcal{L}_{reg}) \cap \alpha'(\mathcal{L}'_{reg})) \subset (\mathfrak{A} \cup \mathfrak{A}')^*$$

- composition:

$$Y \circ X = \{(\alpha \circ \pi_{\mathfrak{A}}(w), \beta' \circ \pi_{\mathfrak{A}'}(w)) \,|\, w \in \hat{\mathcal{L}}\} \subset \mathfrak{B}^* \times \mathfrak{D}^*$$

- for $X(\mathcal{L}) = \beta(\alpha^{-1}(\mathcal{L}) \cap \mathcal{L}_{reg}) \subset \mathcal{L}'$ and $Y(\mathcal{L}') = \beta'(\alpha'^{-1}(\mathcal{L}') \cap \mathcal{L}'_{reg}) \subset \mathcal{L}''$

$$Y(X(\mathcal{L})) = \beta'(\alpha'^{-1}(\beta(\alpha^{-1}(\mathcal{L}) \cap \mathcal{L}_{reg})) \cap \mathcal{L}'_{reg}) =$$

$$(\beta' \circ \pi_{\mathfrak{A}'})((\alpha \circ \pi_{\mathfrak{A}})^{-1}(\mathcal{L}) \cap \hat{\mathcal{L}}) \subset \mathcal{L}''$$

composition of morphisms is associative

- to verify $(X_1 \circ X_2) \circ X_3 = X_1 \circ (X_2 \circ X_3)$
- with $\hat{\mathcal{L}}_{i,i+1} \subset \pi^{-1}_{\mathfrak{A}_i}(\mathcal{L}_i) \cap \pi^{-1}_{\mathfrak{A}_{i+1}}(\mathcal{L}_{i+1})$ and $\hat{\mathcal{L}}_{(12)3} \subset \pi^{-1}_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(\hat{\mathcal{L}}_{12}) \cap \pi^{-1}_{\mathfrak{A}_3}(\mathcal{L}_3)$ and $\hat{\mathcal{L}}_{1(23)} \subset \pi^{-1}_{\mathfrak{A}_1}(\mathcal{L}_1) \cap \pi^{-1}_{\mathfrak{A}_2 \cup \mathfrak{A}_3}(\hat{\mathcal{L}}_{23})$
- have

$$\hat{\mathcal{L}}_{(12)3} = \left\{ w \,\middle|\, \begin{array}{l} \alpha_2 \circ \pi_{\mathfrak{A}_2} \circ \pi_{\mathfrak{A}_2 \cup \mathfrak{A}_3}(w) = \beta_1 \circ \pi_{\mathfrak{A}_1}(w) \\ \beta_2 \circ \pi_{\mathfrak{A}_2}(w) = \alpha_3 \circ \pi_{\mathfrak{A}_3}(w) \end{array} \right\}$$

$$= \left\{ w \,\middle|\, \begin{array}{l} \alpha_2 \circ \pi_{\mathfrak{A}_2}(w) = \beta_1 \circ \pi_{\mathfrak{A}_1} \circ \pi_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(w) \\ \beta_2 \circ \pi_{\mathfrak{A}_2} \circ \pi_{\mathfrak{A}_1 \cup \mathfrak{A}_2}(w) = \alpha_3 \circ \pi_{\mathfrak{A}_3}(w) \end{array} \right\} = \hat{\mathcal{L}}_{1(23)}$$

D0L languages

- triple $\mathcal{D} = (\Sigma, \Phi, \Omega)$ with Σ finite alphabet, $\Phi : \Sigma^* \to \Sigma^*$ monoid homomorphism, initial word $\Omega \in \Sigma^*$

$$\mathcal{L}_{\mathcal{D}} = \{\Phi^n(\Omega) \,|\, n \in \mathbb{N}\}$$

D0L correspondences

- free monoid $\Phi : \mathfrak{B}^* \to \mathfrak{C}^*$
- regular language $\mathcal{L} \subset \mathfrak{B}^*$, context-free language $\mathcal{L}' \subset \mathfrak{C}^*$
- $\Phi(\mathcal{L}) \subset \mathcal{L}'$
- obtain morphism $X_\Phi \in \mathrm{Mor}_{\mathcal{CFL}}(\mathcal{L}, \mathcal{L}')$

$$X_\Phi = \{(w, \Phi(w)) \,|\, w \in \mathcal{L}\} \subset \mathfrak{B}^* \times \mathfrak{C}^*$$

special case

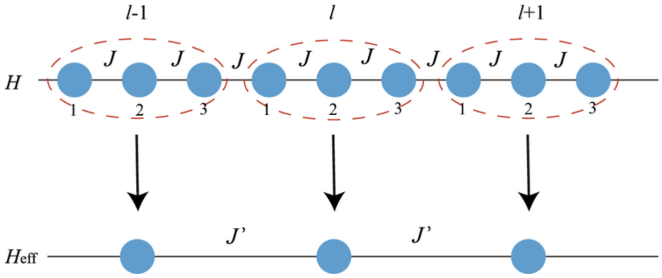
- in particular (for $\mathfrak{B} = \mathfrak{C} = \Sigma$) and $\mathcal{D} = (\Sigma, \Phi, \Omega)$: semigroup

$$\mathcal{X}_\Phi \subset \mathrm{End}_{\mathcal{CFL}}(\Sigma^*), \qquad \mathcal{X}_\Phi = \cup_n X_{\Phi^n}$$

- $\mathcal{L} = \{\Omega\}$ singleton regular language:

$$\mathcal{X}_\Phi(\Omega) = \mathcal{L}_{\mathcal{D}}$$

spin chains



spin chains

- one-dimensional Ising chain, spin variables $\sigma_j = \pm 1$

$$H(\sigma) = -\sum_{j=1}^{N} J_j \sigma_j \sigma_{j+1} - \sum_{j=1}^{N} H_j \, \sigma_j$$

- Boltzmann weights

$$W(\sigma) = \prod_{j=1}^{N} W_j(\sigma_j, \sigma_{j+1}) = e^{-\beta \, H(\sigma)}$$

$$= \prod_{j=1}^{N} e^{\beta (J_j \sigma_j \sigma_{j+1} + \frac{1}{2}(H_j \sigma_j + H_{j+1} \sigma_{j+1}))}$$

- partition function (β inverse temperature)

$$Z_N(\beta) = \sum_{\sigma} e^{-\beta H(\sigma)}$$

transfer matrix

- each site i of spin chain 2×2 *transfer matrix*

$$T_j = \begin{pmatrix} e^{\beta(J_j+\frac{1}{2}(H_j+H_{j+1}))} & e^{-\beta(J_j-\frac{1}{2}(H_j-H_{j+1}))} \\ e^{-\beta(J_j+\frac{1}{2}(H_j-H_{j+1}))} & e^{\beta(J_j-\frac{1}{2}(H_j+H_{j+1}))} \end{pmatrix}$$

- partition function

$$Z_N(\beta) = \mathrm{Tr}(T_1 \cdots T_N) = \mathrm{Tr}(T),$$

with $T = T_1 \cdots T_N$ *monodromy matrix* of spin chain

Kadanoff renormalization

- integrate out degrees of freedom at i-th site (add values $\sigma_i = \pm 1$) while maintaining the same partition function
- change to Hamiltonian
- i-th site factor

$$\exp(K(\sigma_{i-1}\sigma_i + \sigma_i\sigma_{i+1}) + h\sigma_i + \frac{h}{2}(\sigma_{i-1} + \sigma_{i+1})))$$

with $K = \beta J$ and $h = \beta H$ (taking all $J_j = J$ and $H_j = H$)

- summing over $\sigma_i = \pm 1$

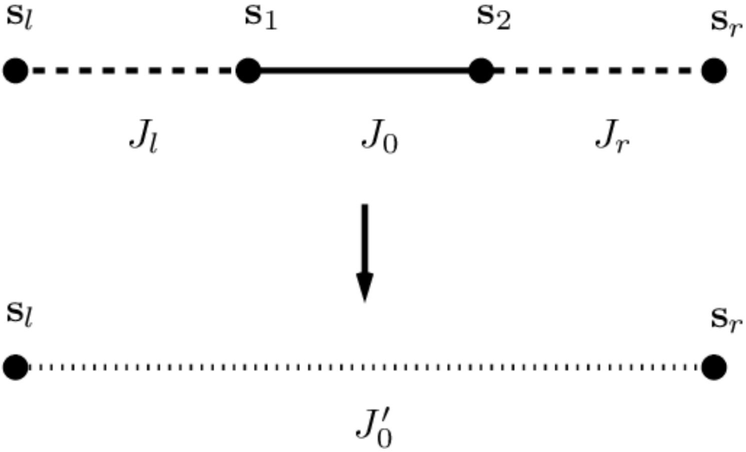
$$\exp((K+\frac{h}{2})(\sigma_{i-1}+\sigma_{i+1})+h)+\exp((-K+\frac{h}{2})(\sigma_{i-1}+\sigma_{i+1})-h)$$

require that this matches

$$\exp(K'\sigma_{i-1}\sigma_{i+1} + \frac{h'}{2}(\sigma_{i-1} + \sigma_{i+1}) + 2g)$$

with $\exp(2g)$ overall non-dynamical factor (indep of σ)

spin decimation



renormalization transformation (spin decimation)

$$K' = \frac{1}{4}\log\frac{\cosh(2K+h)\cosh(2K-h))}{\cosh^2(h)}$$

$$h' = h + \frac{1}{2}\log\frac{\cosh(2K+h)}{\cosh(2K-h)}$$

$$g = \frac{1}{4}\log(\cosh(2K+h)\cosh(2K-h)\cosh(h))$$

- transformation $(K,h)\mapsto(K',h')$
- Hamiltonian $H(\sigma)$ to $H'(\hat\sigma)$ with $\hat\sigma = (\sigma_1,\ldots,\hat\sigma_i,\ldots,\sigma_N)$
- partition function

$$Z_N(\beta) = \sum_\sigma e^{-\beta H(\sigma)} = \sum_{\hat\sigma} e^{-\beta H'(\hat\sigma)} = Z'_{N-1}(\beta)$$

  up to overall multiplicative factor $\exp(2g)$
- view as a transformation (local at single site i)

$$J = \{J_j\}_{j\in\mathcal{I}} \mapsto \mathcal{R}(J) = \{J'_j\}_{j\in\mathcal{I}'}$$

  with $\beta J' = K'$ and $\mathcal{I}' = \mathcal{I}\smallsetminus\{i\}$
- spin decimation in a subset $\mathcal{J}\subset\mathcal{I}$ iteration: $J\mapsto\mathcal{R}(J)=J'$

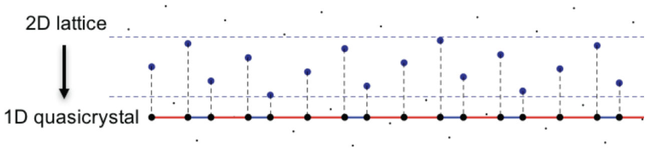
$$J'_{a'} = \mathcal{R}(J_w) \text{ with } w = ab \text{ for } J_{a'} \text{ replacing } J_a, J_b$$

Spin chain sets

- $\Upsilon = (\mathfrak{A}, \Omega, J_{\mathfrak{A}})$ with $\mathfrak{A}$ finite alphabet, $\Omega \subset \mathfrak{A}^*$ nonempty set of words, $J_{\mathfrak{A}} = \{J_a\}_{a\in\mathfrak{A}}$ set of parameters $J_a \geq 0$
- spin chain set S_Υ

$$S_\Upsilon := \sqcup_{w\in\Omega} S_w$$

- S_w spin chain of length $\ell(w)$ (for word w)
- Hamiltonian with $J_j = J_a$ if word w has $w_j = a \in \mathfrak{A}$, and all $H_j = H = 0$



disconnections: some $J_i = 0$ with also $H = 0$, then spin chain decouples into two non-interacting parts and partition function factors as product of two independent subsystems

decompositions and disconnections

- $\alpha : \mathfrak{A}^* \to \mathfrak{B}^*$ homomorphism of free monoids: unique decomposition $\mathfrak{A} = \mathfrak{A}_\alpha \sqcup \mathfrak{A}_{\alpha^\perp}$ with $\alpha(a) \neq \epsilon$ for $a \in \mathfrak{A}_\alpha$ and $\alpha(a) = \epsilon$ for $a \in \mathfrak{A}_{\alpha^\perp}$ (because *free* monoids)
- for any decomposition $\mathfrak{A} = \mathfrak{A}_1 \sqcup \cdots \sqcup \mathfrak{A}_\ell$ of $\mathfrak{A}$ and $\alpha : \mathfrak{A}^* \to \mathfrak{B}^*$ get decomposition $\mathfrak{A} = \bigsqcup_i \mathfrak{A}_{i\alpha} \sqcup \bigsqcup_i \mathfrak{A}_{i\alpha^\perp}$ intersection with $\mathfrak{A}_\alpha \sqcup \mathfrak{A}_{\alpha^\perp}$
- add a new letter to mark disconnection sites: $\mathfrak{A}_o = \mathfrak{A} \cup \{o\}$ and $\alpha_o(o) = \epsilon$
- disconnection $\alpha^{-1}(\Omega)_o$ of set $\Omega \subset \alpha(\mathfrak{A}^*) \subset \mathfrak{B}^*$ along decomposition = subset of $\alpha_o^{-1}(\Omega)$ inserting $uv \mapsto uov$ in words of $\alpha^{-1}(\Omega)$ when u, v in different pieces of decomposition
- disconnection: from the fact that homomorphism α of free monoids can have a kernel

common decimation and disconnection

- spin chain sets S_Υ and $S_{\Upsilon'}$ with $\Upsilon = (\mathfrak{B}, \Omega, J'_{\mathfrak{B}})$ and $\Upsilon' = (\mathfrak{C}, \Omega', J_{\mathfrak{C}})$
- obtain spin chain set $S_{\hat{\Upsilon}}$ with $\hat{\Upsilon} = (\mathfrak{A}_o, \hat{\Omega}, \hat{J}_{\mathfrak{A}_o})$
- with homomorphisms of free monoids $\alpha : \mathfrak{A}^* \to \mathfrak{B}^*$ and $\beta : \mathfrak{A}^* \to \mathfrak{C}^*$
- with $\Omega \subset \alpha(\mathfrak{A}^*)$, $\Omega' \subset \beta(\mathfrak{A}^*)$, $\hat{\Omega} \subset \alpha^{-1}(\Omega)_o$, $\beta_o(\hat{\Omega}) \subset \Omega'$
- for decomposition $\mathfrak{A} = \mathfrak{A}_{\alpha\beta} \sqcup \mathfrak{A}_{\alpha\beta^\perp} \sqcup \mathfrak{A}_{\alpha^\perp\beta} \sqcup \mathfrak{A}_{\alpha^\perp\beta^\perp}$ coming from intersections of decompositions $\mathfrak{A} = \mathfrak{A}_\alpha \sqcup \mathfrak{A}_{\alpha^\perp}$ and $\mathfrak{A} = \mathfrak{A}_\beta \sqcup \mathfrak{A}_{\beta^\perp}$
- with $\hat{J}_o = 0$ and

$$\hat{J}_a = \begin{cases} \mathcal{R}(J_{\alpha(a)}) = \mathcal{R}(J_{\beta(a)}) & a \in \mathfrak{A}_{\alpha\beta} \\ \mathcal{R}(J_{\alpha(a)}) & a \in \mathfrak{A}_{\alpha\beta^\perp} \\ \mathcal{R}(J_{\beta(a)}) & a \in \mathfrak{A}_{\alpha^\perp\beta} \\ 0 & a \in \mathfrak{A}_{\alpha^\perp\beta^\perp}\,. \end{cases}$$

- equality $\mathcal{R}(J_{\alpha(a)}) = \mathcal{R}(J_{\beta(a)})$ for $a \in \mathfrak{A}_{\alpha\beta}$: existence of common decimation

common decimation and disconnection

$$\mathbb{D}(S_{\Upsilon'}, S_{\Upsilon}) = (S_{\hat{\Upsilon}}, \alpha, \beta)$$

where write transformation of Js as

$$\hat{J}_{\mathfrak{A}} = \mathcal{R}_{\alpha}(J_{\mathfrak{B}}) = \mathcal{R}_{\beta}(J_{\mathfrak{C}})$$

- operation of common decimation and disconnection involves:
  1. $\mathfrak{A}_{\alpha\beta}$ contributes a common decimation
  2. $\mathfrak{A}_{\alpha\beta^{\perp}} \sqcup \mathfrak{A}_{\alpha^{\perp}\beta}$ contributes a independent decoupled systems (decoupling by disconnection $\alpha^{-1}(\Omega)_o$)
- combination of decimation and disconnection is needed for good compositional properties

Category of spin chains

- $\mathrm{Obj}(\mathcal{S})$ spin chain sets S_Υ
- $\mathrm{Mor}_{\mathcal{S}}(S_\Upsilon, S_{\Upsilon'})$ common decimations and disconnections of S_Υ and $S_{\Upsilon'}$

composition of morphisms $\mathbb{D}(S_{\Upsilon''}, S_{\Upsilon'}) \circ \mathbb{D}(S_{\Upsilon'}, S_\Upsilon)$

- composition $S_{\hat{\Upsilon}'} \circ S_{\hat{\Upsilon}}$ is

$$\mathbb{D}(S_{\Upsilon''}, S_\Upsilon) = (S_{\hat{\Upsilon}''}, \alpha \circ \pi_{\mathfrak{A}}, \beta' \circ \pi_{\mathfrak{A}'})$$

- with $\hat{\Upsilon}'' = ((\mathfrak{A} \sqcup \mathfrak{A}')_o, \hat{\Omega}'', \hat{J}''_{(\mathfrak{A} \sqcup \mathfrak{A}')_o})$, where $\hat{\Upsilon} = (\mathfrak{A}_o, \hat{\Omega}, \hat{J}_{\mathfrak{A}_o})$ and $\hat{\Upsilon}' = (\mathfrak{A}'_o, \hat{\Omega}', \hat{J}'_{\mathfrak{A}'_o})$
- with $\hat{J}''_o = 0$ and $\hat{\Omega}''$ and $\hat{J}''_{\mathfrak{A} \sqcup \mathfrak{A}'}$

$$\hat{\Omega}'' = \{w \in \pi_{\mathfrak{A}}^{-1}(\hat{\Omega})_o \cap \pi_{\mathfrak{A}'}^{-1}(\hat{\Omega}')_o \,|\, \beta_o(\pi_{\mathfrak{A}}(w)) = \alpha'_o(\pi_{\mathfrak{A}'}(w))\}$$

$$\hat{J}''_a = \hat{J}_a \;\forall a \in \mathfrak{A} \quad \text{and} \quad \hat{J}''_{a'} = \hat{J}'_{a'} \;\forall a' \in \mathfrak{A}'$$

- composition is associative (as for $\mathcal{CFL}$ category)

Modeling spin chains by formal languages

- $\mathcal{CFLJ}$ category with objects $(\mathcal{L}, J)$ with $\mathcal{L} \subset \mathfrak{B}^*$ a context-free language (object in $\mathcal{CFL}$) and $J : \mathfrak{B} \to \mathbb{R}_+^{\#\mathfrak{B}}$ function
- morphisms $(X, \hat{J}) \in \mathrm{Mor}_{\mathcal{CFLJ}}((\mathcal{L}, J), (\mathcal{L}', J'))$, with $\mathcal{L} \subset \mathfrak{B}^*$ and $\mathcal{L}' \subset \mathfrak{C}^*$:
  - pairs of a morphism $X = (\mathfrak{A}, \alpha, \beta, \mathcal{L}_{reg}) \in \mathrm{Mor}_{\mathcal{CFL}}(\mathcal{L}, \mathcal{L}')$ rational transductions
  - and $\hat{J} : \mathfrak{A} \to \mathbb{R}_+^{\#\mathfrak{A}}$ (as in common decimation and disconnection)
- functor $\mathfrak{S} : \mathcal{CFLJ} \to \mathcal{S}$ assigning to a pair $(\mathcal{L}, J)$ spin chain set $\mathfrak{S}(\mathcal{L}) = S_\Upsilon$ with $\Upsilon = (\mathfrak{A}, \mathcal{L}, J_\mathfrak{A})$
- assigning to a morphism $(X, \hat{J}) \in \mathrm{Mor}_{\mathcal{CFLJ}}((\mathcal{L}, J), (\mathcal{L}', J'))$ common decimation and disconnection $\mathbb{D}(S_{\Upsilon'}, S_\Upsilon) = (S_{\hat{\Upsilon}}, \alpha, \beta)$
- for $X = (\mathfrak{A}, \alpha, \beta, \mathcal{L}_{reg})$, identified with $X = (\alpha \times \beta)(\mathcal{L}_{reg}) \subset \mathfrak{B}^* \times \mathfrak{C}^*$, the chain set $S_{\hat{\Upsilon}}$ has

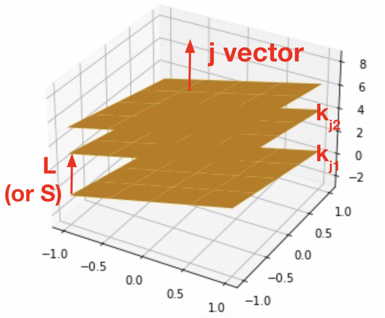
$$\hat{\Upsilon} = (\mathfrak{A}, X \cap (\mathcal{L} \times \mathcal{L}'), \hat{J}_\mathfrak{A}) \quad \text{with} \quad \hat{J}_\mathfrak{A} = \mathcal{R}_\alpha(J_\mathfrak{B}) = \mathcal{R}_\beta(J_\mathfrak{C})$$

Amman quasilattice of icosahedral crystal

- Fibonacci hexagrid set of vectors $\vec{x}$ so that $\vec{x} \cdot \vec{e}_j$ form a 1D Fibonacci quasi-lattice with

$$\vec{e}_j = \left(\frac{2}{\sqrt{5}}\cos\left(\frac{2\pi j}{5}\right), \frac{2}{\sqrt{5}}\sin\left(\frac{2\pi j}{5}\right), \frac{1}{\sqrt{5}}\right) \text{ for } j = 1, \ldots, 4\,, \qquad \vec{e}_5 = (0,0,1)$$

- each plane in the subsystem normal to $\vec{e}_j$ labeled by k_j ordinal position along $\vec{e}_j$ direction



Amman subsystem orthogonal to $\vec{e}_j$, distance (translation vector) $\vec{v}_j$, letter $\mathbb{L}$ or $\mathbb{S}$ spacing between them

- specify the position on this hexagrid in terms of symbols

$$\left[j \, , [\, k_{j_1} \, , \, k_{j_2} \,], \, \vec{v}_j \right]$$

- after substitution rule in the 1D quasilattice in direction $\vec{e}_j$:
    - j unchanged
    - positions $k_{j,1}, k_{j,2}$ unchanged
    - translation vector becomes $(\tau - 1)\vec{v}_j$ after one inflation and $\tau^n(1 - \tau^{-1})\vec{v}_j$ after n inflations
- how to parameterize by formal languages taking into account substitutions in each of the Amman planes subsystems and incidence relations between them?
- a single system in isolation is like a 1D quasilattice spin chain (CF $\mathcal{L}_{\rm Fib}$ and D0L transductions)
- when combined? like several CF grammars in interaction...

Multiple Context Free Grammars (MCFG)

- generalization of context free (mildly context sensitive)
- m-MCFG: set $\mathcal{G} = (\mathfrak{A}, \mathcal{N}, P, S)$ with
  - alphabet $\mathfrak{A}$ of terminal symbols
  - set $\mathcal{N}$ of nonterminals
  - each $N \in \mathcal{N}$ has a dimension $1 \leq d(N) \leq m$
  - initial nonterminal $S \in \mathcal{N}$ with $d(S) = 1$
  - set P production rules:

    $$A \mapsto f(B_1, \ldots, B_q) \quad \text{or} \quad A \mapsto (w_1, \ldots, w_{d(A)})$$

    $w_k \in \mathfrak{A}^*$ are strings of terminals and $f : \mathcal{N}^q \to \mathcal{N}^{d(A)}$ functions with $f = (f_1, \ldots, f_{d(A)})$
  - each $\omega_k = f_k(B_1, \ldots, B_q)$ string involving terminal symbols and components $x_{j,r}$ of a subset of nonterminals $B_j = (x_{j,r})_{r=1}^{d(B_j)}$, each B_j occurring at most once in ω_k
- language $\mathcal{L}_{\mathcal{G}}$: all the strings of terminals obtained starting from S by repeated application of production rules
- derivation $A \xrightarrow{\bullet} w$ tuple of strings $w \in (\mathfrak{A}^*)^{d(A)}$ obtained by repeated production from nonterminal $A \in \mathcal{N}$
- $\mathcal{L}_{\mathcal{G}} = \{ w \in \mathfrak{A}^* \,|\, S \xrightarrow{\bullet} w \}$

Example: 2-MCFG

- $\mathfrak{A} = \{a, b, c, d\}$ and $\mathcal{N} = \{S, N, M\}$ with $d(N) = d(M) = 2$
- production rules

$$S \mapsto f(N, M), \quad N \mapsto (a, c), \quad M \mapsto (b, d),$$

$$N \mapsto g(N), \quad M \mapsto h(M)$$

- with $f((x_1, x_2), (y_1, y_2)) = x_1 y_1 x_2 y_2$, $g(x_1, x_2) = (x_1 a, x_2 c)$, and $h(y_1, y_2) = (y_1 b, y_2 d)$
- computes language $\mathcal{L}_{\mathcal{G}} = \{a^n b^m c^n d^m \,|\, n, m \in \mathbb{N}\}$
- contains cross-serial dependences so not context-free

functional description

- describe m-MCFG $\mathcal{G}$ by set of functions $F_{\mathcal{G}}$ in production rules and set $\Sigma_{\mathcal{G}}$ of arguments so $\mathcal{L}_{\mathcal{G}} = F_{\mathcal{G}}(\Sigma_{\mathcal{G}})$
- tuples $(\omega_1, \ldots, \omega_q) \in \Sigma_{\mathcal{G}}$, k-th component $\omega_k \in (\mathfrak{A}^*)^{d(B_k)}$, with $B_k \xrightarrow{\bullet} \omega_k$, for $f(B_1, \ldots, B_q) \in F_{\mathcal{G}}$

Category of m-MCFGs

- objects of $\mathcal{MCFG}_m$ are the m-MCFGs $\mathcal{G} = (\mathfrak{A}, \mathcal{N}, P, S)$
- morphisms $\mathrm{Mor}_{\mathcal{MCFG}_m}(\mathcal{G}, \mathcal{G}')$ matrices of rational transductions $X = (X_{k\ell})_{k,\ell=1}^{q}$
- with $X_{k\ell} : (\mathfrak{A}^*)^{d(B_k)} \to (\mathfrak{B}^*)^{d(B'_\ell)}$ such that $X_{k\ell}(\Sigma_{\mathcal{G},k}) \subset \Sigma_{\mathcal{G}',\ell}$
- such that for any $f' \in F_{\mathcal{G}'}$ the precomposition $f = f' \circ X$ is in $F_{\mathcal{G}}$
- and such that all the $X_{k,\ell}$ have the same $\mathcal{L}_{reg}$

MCFGs and the Amman quasilattice

- data $\mathcal{G} = (\mathfrak{A}, \mathcal{N}, P, S)$ with $\mathfrak{A} = \{\mathbb{L}_j, \mathbb{S}_j\}_{j=0}^{5}$

$$\mathcal{N} = \{S, M, M_j, N_j, A_j, B_j \,|\, j = 0, \ldots, 5\}$$

with $d(M) = 3$, $d(M_j) = d(N_j) = d(A_j) = d(B_j) = 1$, $M = (x, y, z)$

- production rules

$$S \to f(M)\,, \quad M \to g(M, M_j)\,, \quad M \to h_{j_1,j_2,j_3}(N_0, \ldots, N_5)\,,$$

with $f(x, y, z) = xyz$, $g((x, y, z), u) = (xu, yu, zu)$ and

$$h_{j_1,j_2,j_3}(x_0, \ldots, x_5) = (x_{j_1}, x_{j_2}, x_{j_3})\,,$$

indexed by triples (j_1, j_2, j_3) of distinct indices $j_k \in \{0, \ldots, 5\}$, and

$$N_j \to A_j\,, \quad N_j \to B_j$$
$$A_j \to A_j B_j\,, \quad B_j \to A_j$$
$$M_j \to \mathbb{L}_j\,, \quad M_j \to \mathbb{S}_j$$
$$A_j \to \mathbb{L}_j\,, \quad B_j \to \mathbb{S}_j$$

- this determines a MCFG with

$$\Sigma_{\mathcal{G}} = \{(w_{j_1}\omega_j, w_{j_2}\omega_j, w_{j_3}\omega_j) \,|\, w_{j_k} \in \mathcal{L}_{\rm Fib}\,, \omega_j \in \{\mathbb{S}_j, \mathbb{L}_j\}\}$$

  for any triple (j_1, j_2, j_3) of distinct indices $j_k \in \{0, \ldots, 5\}$ and any $j \in \{0, \ldots, 5\} \smallsetminus \{j_1, j_2, j_3\}$
- with $\mathcal{L}_{\rm Fib}$ the Fibonacci language
- words in the resulting language $\mathcal{L}_{\mathcal{G}}$ parameterize regions of the Ammann quasilattice
- D0L substitution ϕ defines an endomorphism of this MCFG in the category $\mathcal{MCFG}_3$
- diagonal morphism $X = (X_{ab})_{a,b=1}^3$ acting on triples as

$$\phi : (w_{j_1}, w_{j_2}, w_{j_3}) \mapsto (\phi(w_{j_1}), \phi(w_{j_2}), \phi(w_{j_3})) \quad \text{gives}$$

$$\phi : (w_{j_1}\omega_j, w_{j_2}\omega_j, w_{j_3}\omega_j) \mapsto (\phi(w_{j_1})\omega_{j,\phi}, \phi(w_{j_2})\omega_{j,\phi}, \phi(w_{j_3})\omega_{j,\phi})$$

  and correspondence (as graph)

$$X_\phi = \{((w_{j_1}\omega_j, w_{j_2}\omega_j, w_{j_3}\omega_j), (\phi(w_{j_1})\omega_{j,\phi}, \phi(w_{j_2})\omega_{j,\phi}, \phi(w_{j_3})\omega_{j,\phi})\}$$

  (no longer diagonal, since $\omega_{j,\phi}$ depends also on $(w_{j_1}, w_{j_2}, w_{j_3})$ not only on ω_j)

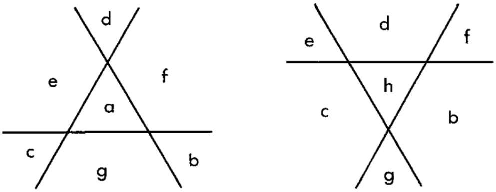
- in terms of symbols $\Big[j\,, [\, k_{j_1}\,,\, k_{j_2}\,],\, \vec{v}_j \Big]$
- for $w = \mathbb{L}$ or $\mathbb{S}$

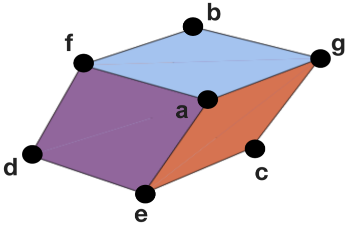
$$\Phi(w) \Big[j\,, [\, k_{j_1}\,,\, k_{j_2}\,],\, \vec{v}_j \Big] = \phi(w) \Big[j\,, [\, k_{j_1}\,,\, k_{j_2}\,],\, (\tau - 1)\vec{v}_j \Big]$$

- for multiple inflations n, we have (for $w = \mathbb{L}$ or $\mathbb{S}$)

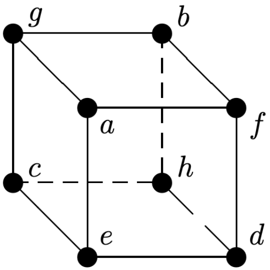
$$\Phi^n(w) \Big[j\,, [\, k_{j_1}\,,\, k_{j_2}\,],\, \vec{v}_j \Big] = \phi^n(w) \Big[j\,, [\, k_{j_1}\,,\, k_{j_2}\,],\, \tau^n (1 - \frac{1}{\tau})\vec{v}_j \Big]$$

Icosahedral quasicrystal and spins on cubes





eight regions cut out by a parallel pair of j-hyperplanes cutting near a triple intersection of $\{j_1, j_2, j_3\}$ hyperplanes in the Ammann quasilattice, and corresponding eight vertices of a rhombohedron



$$= \quad W(a|e,f,g|b,c,d|h)$$

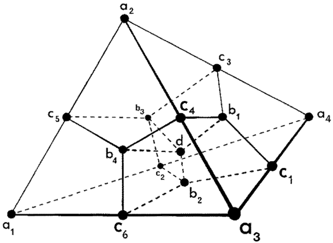
Boltzmann weights for spins at vertices of rhombohedron dual to the eight regions cut out by a triple j_1, j_2, j_3-intersection and a pair of j-hyperplanes in the Ammann quasi-lattice

Korepin's completely integrable model on the icosahedral quasicrystal

- V.E. Korepin. *Completely integrable models in quasicrystals*. Communications in Mathematical Physics 110 (1987), 157–171
- spins on vertices of dual rhombohedra satisfy Zamolodchikov equation
- describes partition function of four skewed rhombohedra joined together with a common interior spin d to form a rhombic dodecahedron
- identity under symmetry with respect to reflection about center of barycentric subdivision of the 3-simplex

Zamolodchikov tetrahedron equation

fourteen "external" spins $a_1, \ldots, a_4, b_1, \ldots, b_4, c_1, \ldots c_6$, and one "internal" spin d



$$\sum_d W(a_4|c_2c_1c_3|b_1b_3b_2|d)W'(c_1|b_2a_3b_1|c_4dc_6|b_4)$$
$$W''(b_1|dc_4c_3|a_2b_3b_4|c_5)W'''(d|b_2b_4b_3|c_5c_2c_6|a_1)$$
$$= \sum_b W'''(b_1|c_1c_4c_3|a_2a_4a_3|d)W''(c_1|b_2a_3a_4|dc_2c_6|a_1)$$
$$W'(a_4|c_2dc_3|a_2b_3a_1|c_5)W(d|a_1a_3a_2|c_4c_5c_6|b_4)$$

connecting d to $a_1, \ldots, a_4$ or $b_1, \ldots b_4$ with $a_i \leftrightarrow b_i$ and $c_1 \leftrightarrow c_5$, $c_2 \leftrightarrow c_4$ and $c_3 \leftrightarrow c_6$

Spin systems on cubes

- k-tuples of words $\Upsilon^{(k)} = (\mathfrak{A}, \Omega^{(k)}, \underline{J}_{\mathfrak{A}^k})$ with $\Omega^{(k)} \subset (\mathfrak{A}^*)^k$
- spin system

$$S_{\Upsilon^{(k)}} = \bigsqcup_{\omega=(\omega_1,\ldots,\omega_k)\in\Omega^{(k)}} S_\omega$$

- S_ω spin system with interacting spins located at the 2^k vertices of a k-dimensional cube
- spins $\sigma_{a_{1,i_1},\ldots,a_{k,i_k},\epsilon_1,\ldots,\epsilon_k}$, for $\omega_i = a_{i_0}\cdots a_{i_{\ell(\omega_i)}}$ and with $\epsilon_i = \{\pm 1\}$
- cube center labelled by $(a_{1,i_1},\ldots,a_{k,i_k})$
- identifications

$$\sigma_{a_{1,i_1},\ldots,a_{i,j},\ldots,a_{k,i_k},\epsilon_1,\ldots,\underbrace{+}_{\text{i-th}},\ldots\epsilon_k} = \sigma_{a_{1,i_1},\ldots,a_{i,j+1},\ldots,a_{k,i_k},\epsilon_1,\ldots,\underbrace{-}_{\text{i-th}},\ldots\epsilon_k}$$

- interaction: Boltzmann weights

$$W^{(k)}(\sigma_{\underline{\epsilon}}) = W(\sigma_{\underline{\epsilon}(0)}|\sigma_{\underline{\epsilon}(1)}|\cdots|\sigma_{\underline{\epsilon}(k-1)}|\sigma_{\underline{\epsilon}(k)})$$

- $\underline{\epsilon}(j)$ vector in $\{\pm 1\}^k$ with j entry equal to -1
- $\sigma_A := (\sigma_a)_{a \in A}$ for a subset $A \subset \{\pm 1\}^k$
- coupling parameters: $\underline{J}_{\mathfrak{A}^k}$ set of vectors of all couplings in Boltzmann weights: $\underline{J}_{a_1,\dots,a_k} \in \underline{J}_{\mathfrak{A}^k}$ vector of interaction parameters at $(a_{1,i_1}, \dots, a_{k,i_k})$ locations
- no need for now of specific assumptions on form of Boltzmann weights: vertex models

block spin renormalization

- spin system spins assigned to cubes labelled by words $\omega \in (\mathfrak{B}^*)^k$ and interaction couplings depending on parameters $\underline{J}_{a_1,\ldots,a_k} \in \underline{J}_{\mathfrak{B}^k}$
- admits a block spin renormalization if there is injective homomorphism $\alpha : \mathfrak{A}^* \to \mathfrak{B}^*$ of free monoids and new set of parameters $\hat{\underline{J}}_{\mathfrak{A}^k}$ with

$$W^{(k)}_{\underline{J}_{\mathfrak{B}^k}}(\sigma_{\omega=\alpha(\eta)}) = W^{(k)}_{\hat{\underline{J}}_{\mathfrak{A}^k}}(\sigma_\eta)$$

- write transformation as

$$\hat{\underline{J}}_{\mathfrak{A}^k} = \mathcal{R}_\alpha(\underline{J}_{\mathfrak{B}^k})$$

- Note: if homomorphism has a kernel, get combination of decimation and disconnection

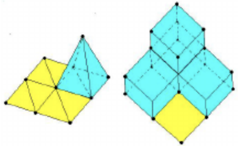
Spin systems from MCFGs functorially

- category $\mathcal{MCFG}_m\mathcal{J}$:
  - objects pairs $(\mathcal{G}, J)$ with $\mathcal{G} \in \mathrm{Obj}(\mathcal{MCFG}_m)$ and $J : \mathfrak{A}^m \to \mathbb{R}^{m\cdot N\cdot\#\mathfrak{A}}$, some $N \in \mathbb{N}$, with $\underline{J}_{\mathfrak{A}^m} := J(\mathfrak{A}^m)$
  - morphisms $(X, \hat{J})$ in $\mathrm{Mor}_{\mathcal{MCFG}_m\mathcal{J}}((\mathcal{G}, J), (\mathcal{G}', J'))$ are morphisms X in $\mathrm{Mor}_{\mathcal{MCFG}_m}(\mathcal{G}, \mathcal{G}')$ with $\hat{J} : \mathfrak{A}^m \to \mathbb{R}^{m\cdot N\cdot\#\mathfrak{A}}$ with $\hat{J} = \mathcal{R}(J) = \mathcal{R}(J')$
- functor $\mathfrak{S}(\mathcal{G}, J) = S_{\Upsilon^{(m)}}$ with $\mathcal{G} = (\mathfrak{A}, \mathcal{N}, P, S)$
- with $\Sigma_{\mathcal{G}}$ and $F_{\mathcal{G}}$ of the MCFG and $\Upsilon^{(m)} = (\mathfrak{A}, \Sigma_{\mathcal{G}}, \underline{J}_{\mathfrak{A}^m})$
- on morphisms: $\mathfrak{S}(X, \hat{J}) = (\mathbb{D}(\Upsilon^{(m)\prime}, \Upsilon^{(m)}), \hat{J})$, with $\mathbb{D}(\Upsilon^{(m)\prime}, \Upsilon^{(m)})$ (common decimation/disconnection)
  - triple $(S_{\hat{\Upsilon}^{(k\times k)}}, \underline{\alpha}, \underline{\beta})$ with $\underline{\alpha} = (\alpha_{i,j}) : (\mathfrak{A}^*)^k \to (\mathfrak{B}^*)^k$ and $\underline{\beta} = (\beta_{ij}) : (\mathfrak{A}^*)^k \to (\mathfrak{C}^*)^k$
  - $S_{\hat{\Upsilon}^{(k\times k)}} = \sqcup_{\eta=(\eta_i)_i \in \hat{\Omega}^{(k\times k)}} S_{\eta_i}$ with $\hat{\Upsilon}^{(k\times k)} = (\mathfrak{A}_o, \hat{\Omega}^{(k\times k)}, \underline{\hat{J}}_{\mathfrak{A}_o^k})$ and $\hat{\Omega}^{(k\times k)} = (\underline{\alpha} \times \underline{\beta})(\mathcal{L}^k_{reg}) \cap (\Omega^{(k)} \times \Omega^{(k)\prime})$

- functor $\mathfrak{S}$ is construction of spin systems associated to multiple context-free grammars
- in $\mathcal{MCFG}_m\mathcal{J}$ category, assignment of parameters $J : \mathfrak{A}^m \to \mathbb{R}^{m \cdot N \cdot \# \mathfrak{A}}$ determines Boltzmann weights of spin system, while MCFG determines sites (locations in Amman quasilattice system)
- image under $\mathfrak{S}$ in category of spin systems gives vertex systems on hypercubes
- what happens to Zamolodchikov tetrahedron equation in this general setting?

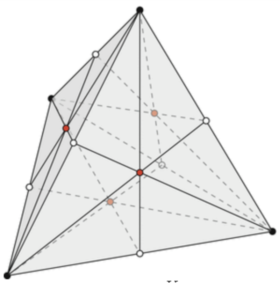
cubulations of the n-simplex

- cubulation of a topological space is a decomposition into a union of cubes intersecting along faces (like triangulations but for cubical complexes instead of simplicial complexes)

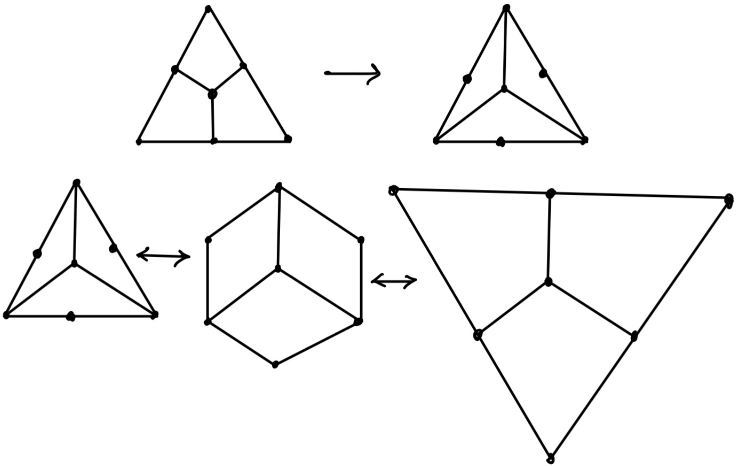


- there is a dual pair of cubulations $\mathcal{C}(\Delta_n)$ and $\mathcal{C}(\Delta'_n)$ of the n-simplex Δ_n
- both consisting of $(n+1)$ cubes of dimension n

- start with first barycentric subdivision $\mathcal{B}(\Delta_n)$ of Δ_n



- in $\mathcal{C}(\Delta_n)$ take each cube as union of all simplexes of $\mathcal{B}(\Delta_n)$ incident to same vertex of Δ_n
- in $\mathcal{C}(\Delta'_n)$ take each cube as union of all simplexes of $\mathcal{B}(\Delta_n)$ incident to same middle vertex of $\mathcal{B}(\Delta_n)$ inside one of the $(n-1)$-faces of Δ_n



the two dual cubulations $\mathcal{C}(\Delta_2)$ and $\mathcal{C}(\Delta_2')$ of Δ_2

paths in the cubes

- n-dim cube $C = [0,1]^n$ has n independent paths of edges starting at $v_0 = (0, 0 \ldots, 0)$ ending at $v_n = (1, 1 \ldots, 1)$ and no other vertex in common
- in cubulation of Δ_n parameterize the cubes so that the vertex common to all cubes is $(1, 1 \ldots, 1)$ in each cube: then these n paths in each C are incident to all $n + 1$ cubes of cubulation
- take the two cubulations $\mathcal{C}(\Delta_n)$ and $\mathcal{C}(\Delta'_n)$, each with its set of independent paths constructed in this way
- call $v^i_{k,r}$ and $\hat{v}^i_{k,r}$ the vertices of the independent paths γ^i_k in $\mathcal{C}(\Delta_n)$ and $\hat{\gamma}^i_k$ in $\mathcal{C}(\Delta'_n)$ (k-th cube, r-th vertex of path)
- vertices $\hat{v}^i_{k,r}$ opposite vertices of the $v^i_{k,r}$ with respect to center of barycentric subdivision

Boltzmann weights on cubulations

- consider combinations of Boltzmann weights along the independent paths

$$\sum_v \prod_{k=0}^{n} W_{(k)}(\sigma_{v_{k,0}} | (\sigma_{v^i_{k,1}})_{i=1}^n | \cdots | (\sigma_{v^i_{k,n-1}})_{i=1}^n | \sigma_{v_{k,n}})$$

$v =$ where all the $(n+1)$-cubes of the cubulation meet (in each simplex, sum over simplexes)

- simplex cubulation generalization of Zamolodchikov equation:

$$\sum_v \prod_{k=0}^{n} W_{(k)}(\sigma_{v_{k,0}} | (\sigma_{v^i_{k,1}})_{i=1}^n | \cdots | (\sigma_{v^i_{k,n-1}})_{i=1}^n | \sigma_{v_{k,n}}) =$$

$$\sum_v \prod_{k=0}^{n} W_{(k)}(\sigma_{\hat{v}_{k,0}} | (\sigma_{\hat{v}^i_{k,1}})_{i=1}^n | \cdots | (\sigma_{\hat{v}^i_{k,n-1}})_{i=1}^n | \sigma_{\hat{v}_{k,n}})$$

Conclusions

1. description of icosahedral quasicrystal by system of Amman planes quasilattice generalizes to coding of quasilattice locations by multiple context-free grammars
2. intersection patterns of Amman planes giving vertex model of spins on cubes generalizes to functorial construction of spin systems on hypercubes from data of multiple context-free grammars and parameters for Boltzmann weights, with common decimations (inflation rules) as morphisms
3. Zamolodchikov tetrahedron equation for Korepin exactly solvable fully integrable system on the icosahedral quasicrystal generalizes to dual simplex cubulations Zamolodchikov-type equation on spin systems constructed from multiple context-free grammars

further directions

- same categorical construction of category of formal languages with rational transductions provides functorial parameterization of 1D Boolean TQFTs with defects
  - Boateng, Luisa; Marcolli, Matilde, *Formal languages and TQFTs with defects*. J. Geom. Phys. 222 (2026), Paper No. 105771, 22 pp.
- fractal boundary in tensor network models of AdS/CFT on Bourdon hyperbolic Bruhat-Tits buildings
  - Gesteau, Elliott; Marcolli, Matilde; Parikh, Sarthak Holographic tensor networks from hyperbolic buildings. J. High Energy Phys. 2022, no. 10, Paper No. 169, 28 pp.
- conformal quasicrystals, holography, and the category of formal languages (project with Alyna Tang)
- spin systems at the boundary of Bourdon buildings (project with Giuseppe di Giulio, et al.)