The Penrose Tiling is a Quantum Error-Correcting Code

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What is this talk (not) about?

1. Direct connection

Penrose tilings → quantum code

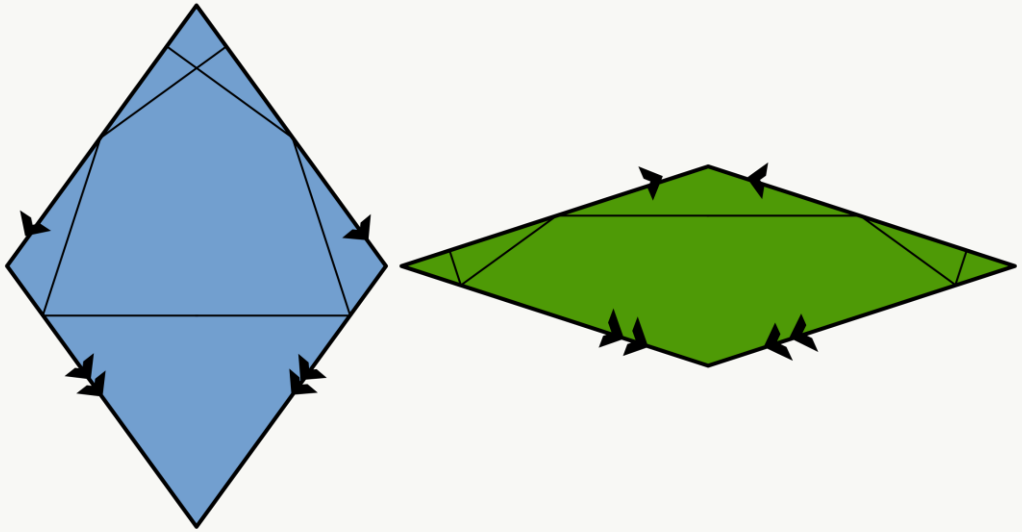
2. Penrose tiling is an example

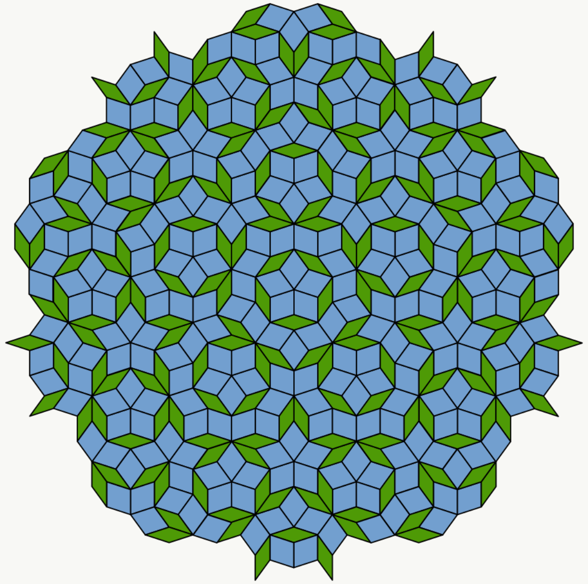
Other quasicrystals (Fibonacci, Ammann–Beenker, etc) also work

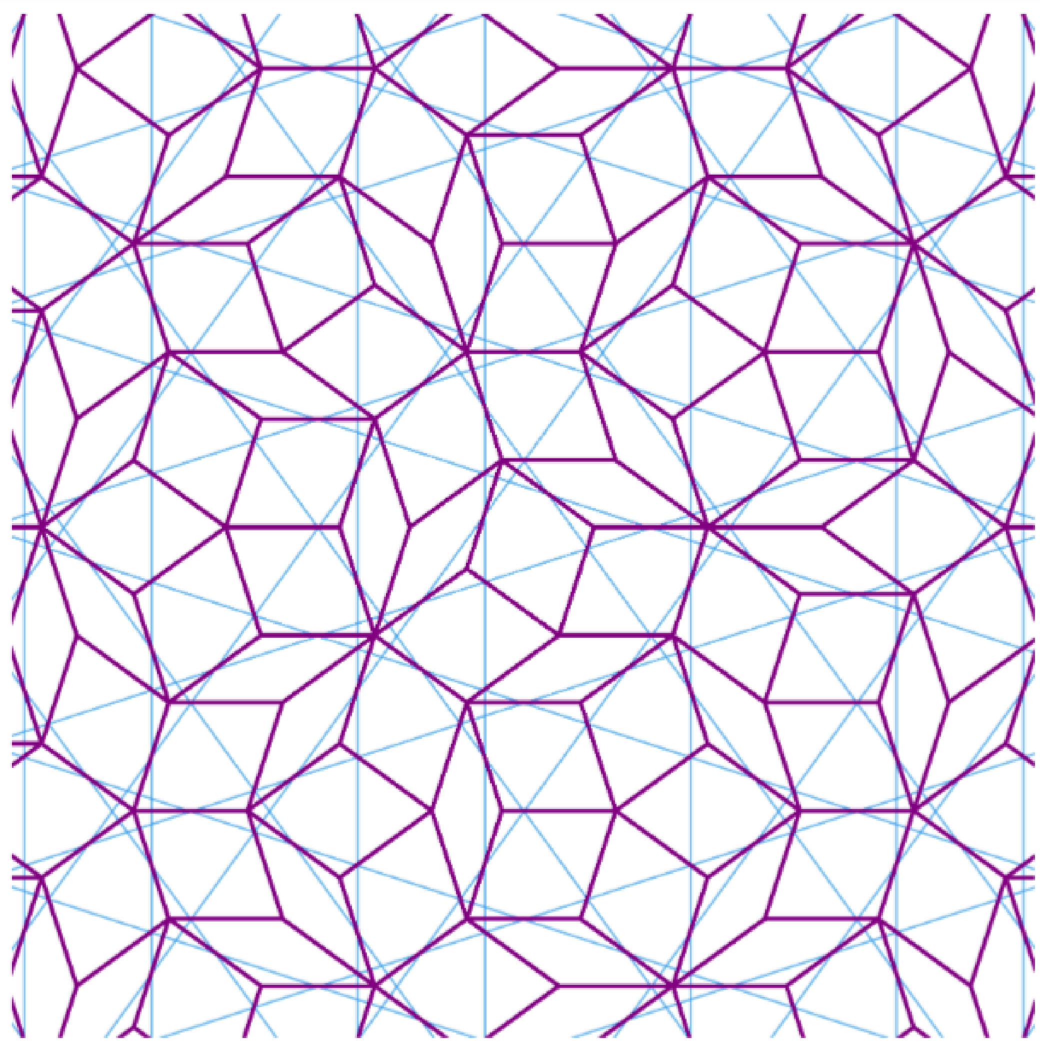
3. Not about physical system defined on quasicrystal lattices

Instead, quantum information is encoded in quantum geometry

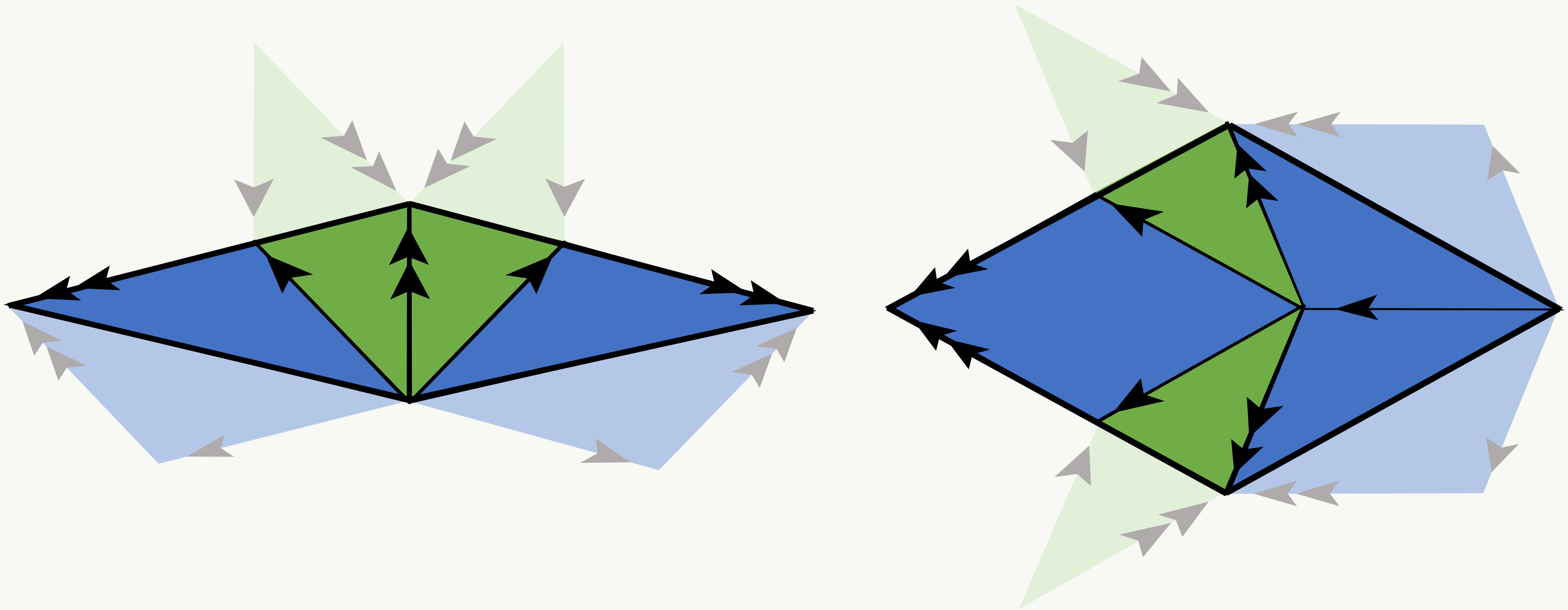
Penrose tilings: matching rule





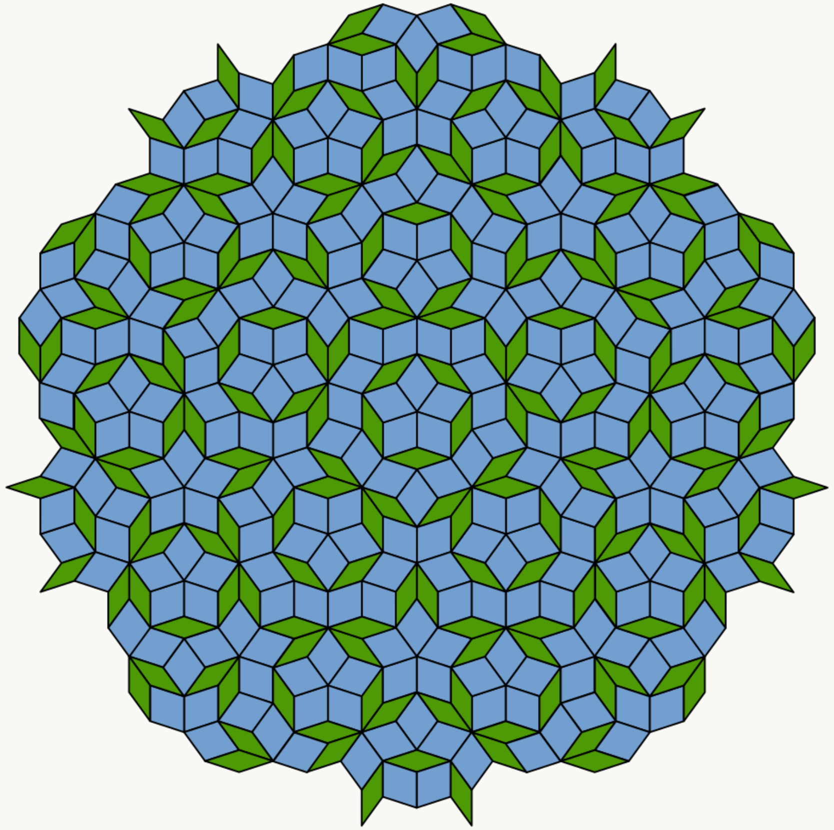


Penrose tiling: Inflation and Deflation

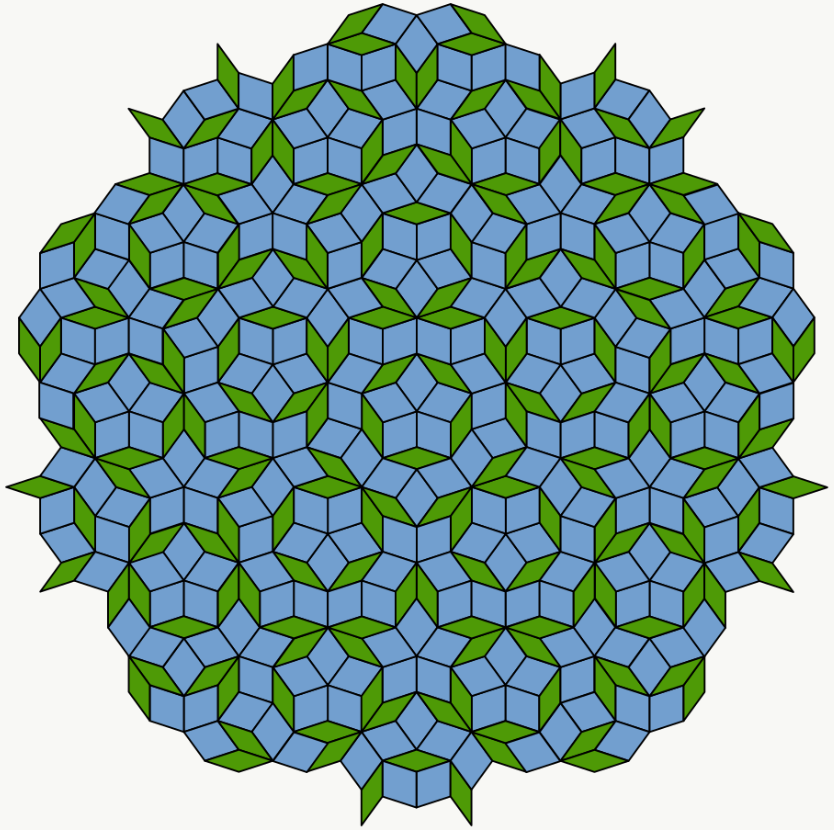


- Inflation: rescale by 1/golden ratio and subdivide
- Can be iterated indefinitely in both directions.
- Deflation: unique grouping into supertiles.

Uncountably many tilings



One Penrose tiling



Another Penrose tiling

Local indistinguishability

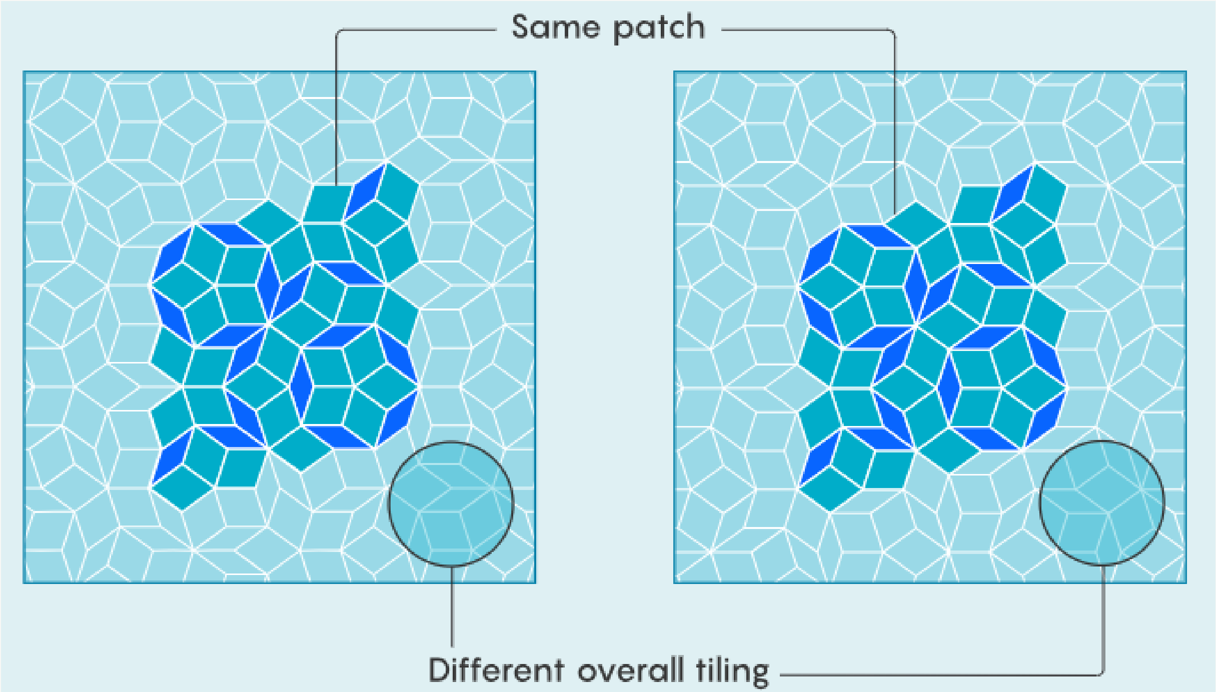
For any finite patch P and any two Penrose tilings T,T'

$$\text{P occurs in T} \Leftrightarrow \text{P occurs in T}'$$

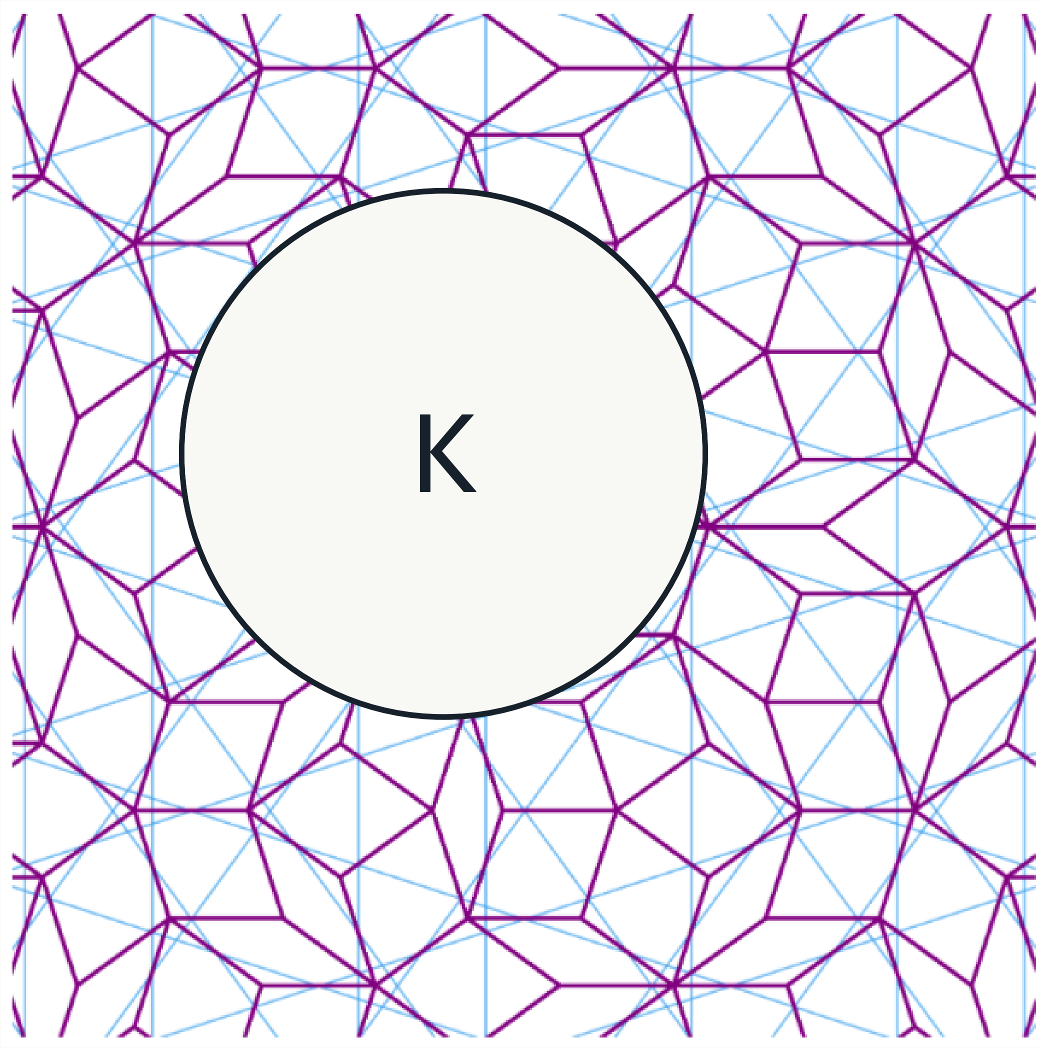
$$\text{freq}_{\text{T}}(\text{P}) = \text{freq}_{\text{T}'}(\text{P})$$

- Same finite patches in every tiling T
- Same relative frequencies

- No local observable can distinguish T and T’
- Differences are global



Recovery from the exterior



If we only know T on K^c:

1 Read the Ammann lines in K^c

directions and offsets are fixed outside

2 Continue each straight line through K

3 Reconstruct the rhombi

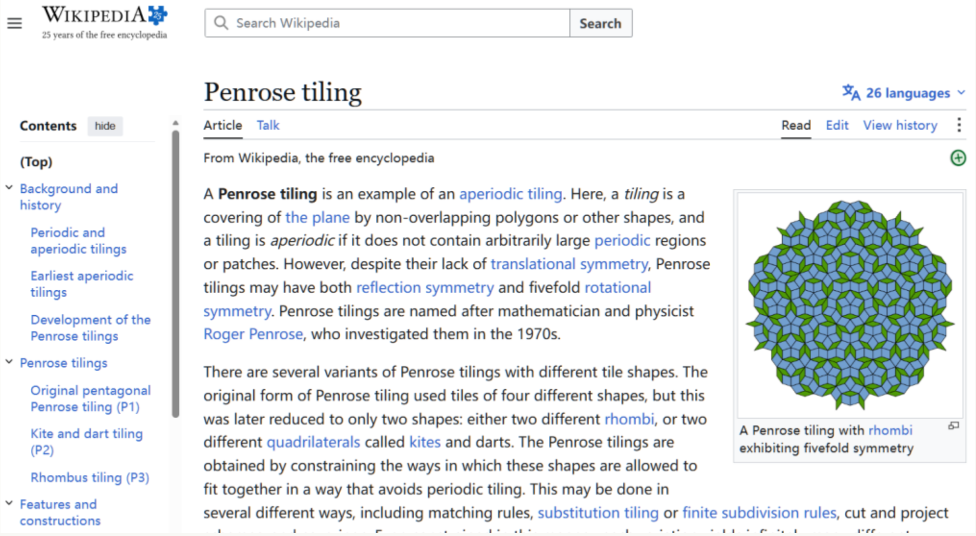
the Ammann pattern uniquely fixes the PT

$$T|_{K^c} = T'|_{K^c} \Rightarrow T = T'$$

(For any finite K)

The Penrose Tiling is a Quantum Error-Correcting Code

A classical code you already use: QR code



Simplest classical code: repetition

Classical repetition code

- Repeat each symbol.
- Allow deletions or bit flips.
- Majority vote to decode

III aaammm

aaa cccooodddeee

redundant representation

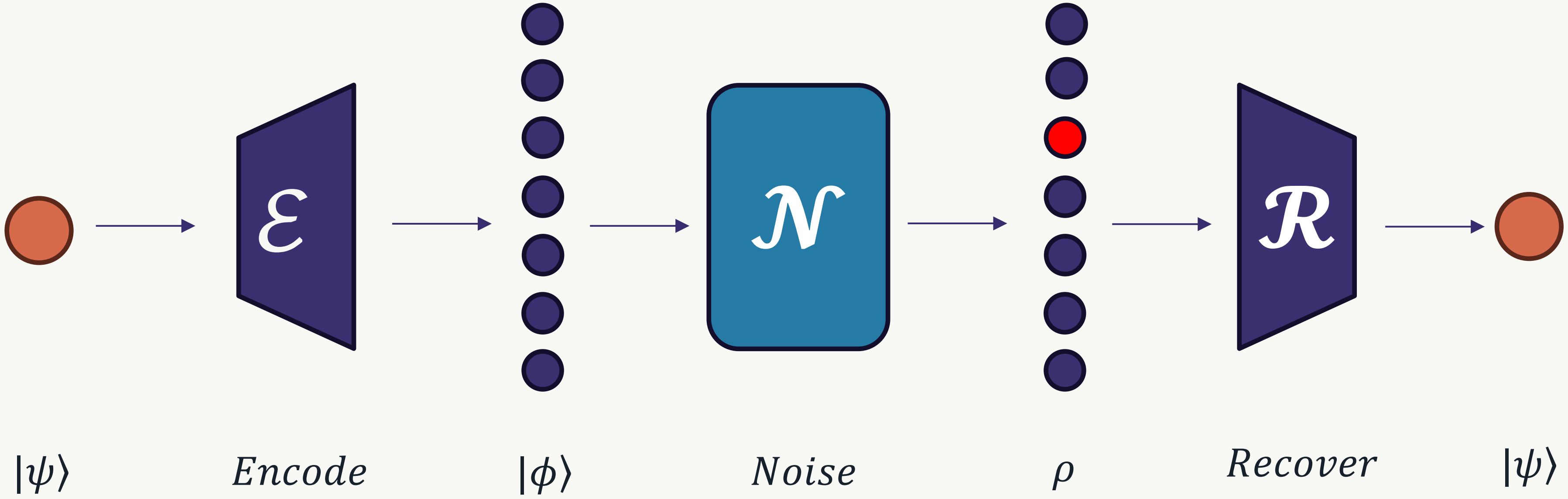
I am a code

encoded message

Error-correcting Code: Protect information by redundancy

Quantum error correction

QEC: $\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}(|\psi\rangle) = |\psi\rangle$



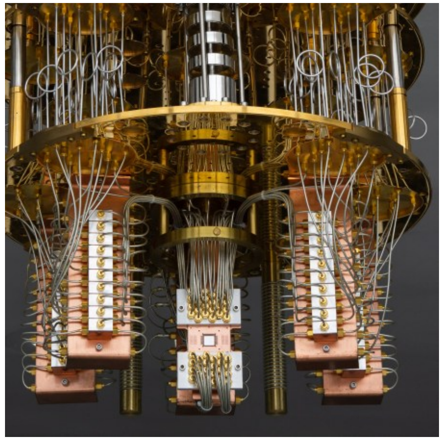
Protect an arbitrary unknown state — including every superposition.

Preserve phase coherence and entanglement with external systems.

Repetition doesn't work: no-cloning.

Usage of QEC

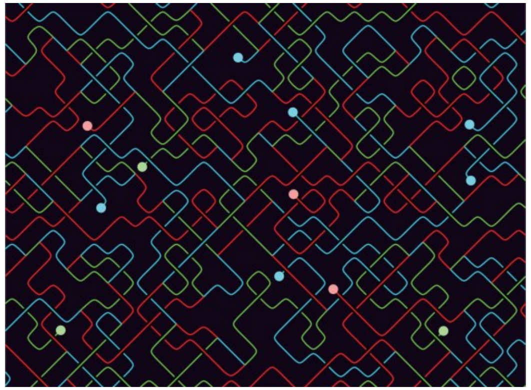
FAULT TOLERANCE



Reliable computation

Logical gates survive imperfect hardware below a threshold.

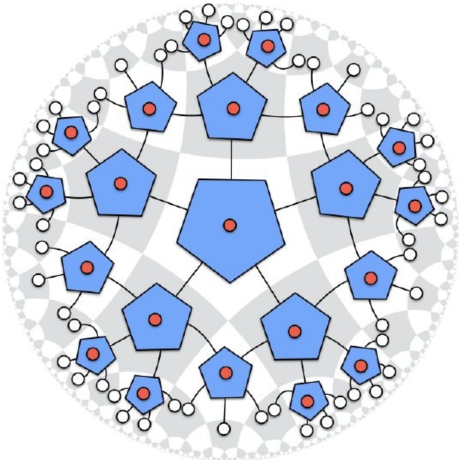
TOPOLOGICAL ORDER



Ground space as a code

Local operators cannot read the global topological sector.

HOLOGRAPHY

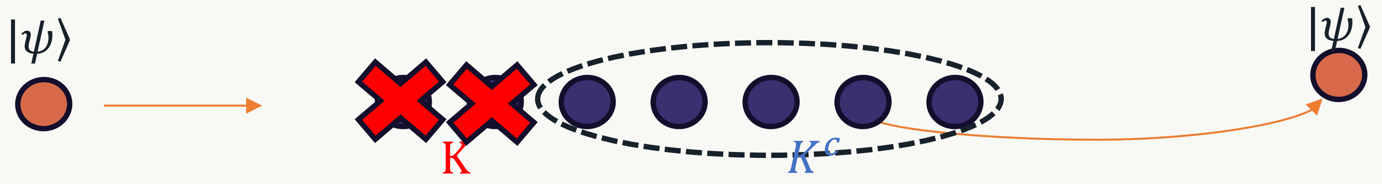


Bulk-to-boundary encoding

Bulk operators reconstructed from boundary subregions.

Pastawski et al. (2015)

Quantum error-correcting code

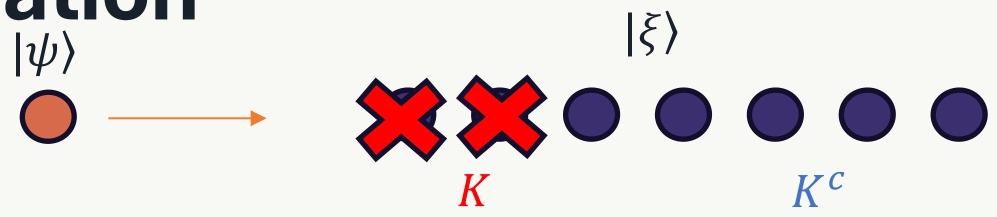


$|K| <$ **code distance $\boldsymbol{d}$**

No Cloning: full information in K^c $\leftrightarrow$ no information in K

[*for experts: monogamy of entanglement]

Correctable erasure ⇔ no information



Erased region K, decompose $\mathcal{H} = \mathcal{H}_K \otimes \mathcal{H}_{K^c}$

Knill-Laflamme condition on code space $\mathcal{C}$:

$$\boxed{\rho_K(\xi) = \mathrm{Tr}_{K^c}(|\xi\rangle\langle\xi|) \text{ independent of } |\xi\rangle \in \mathcal{C}}$$

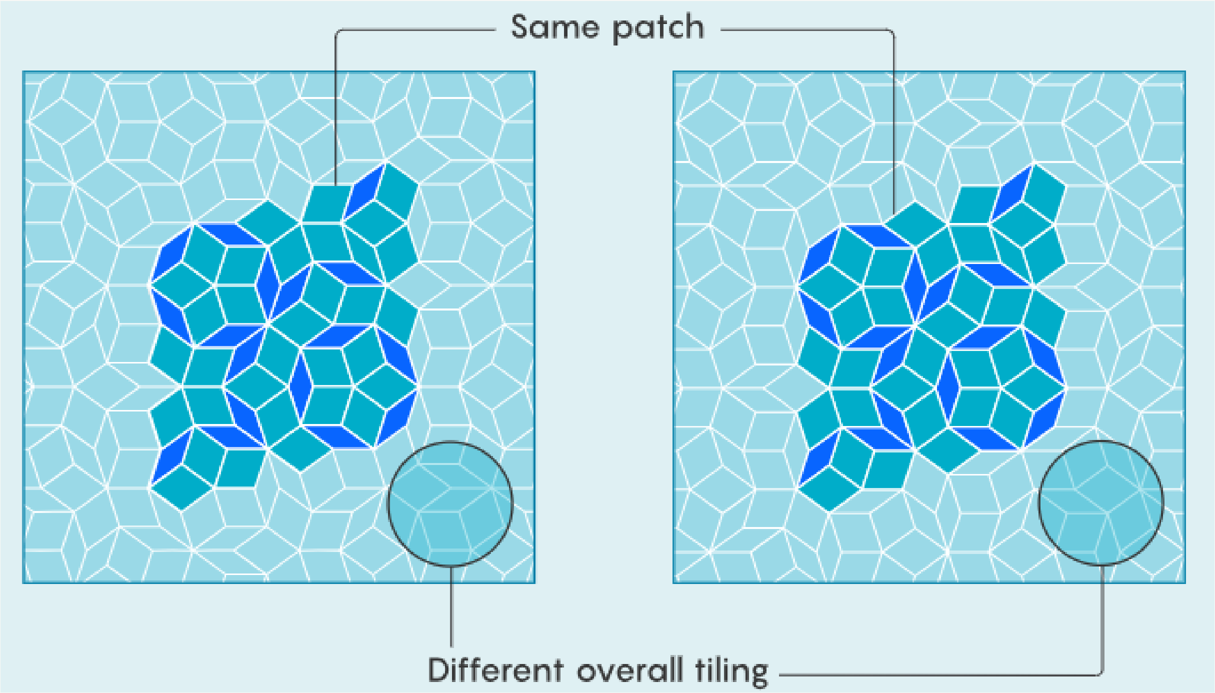
Local indistinguishability

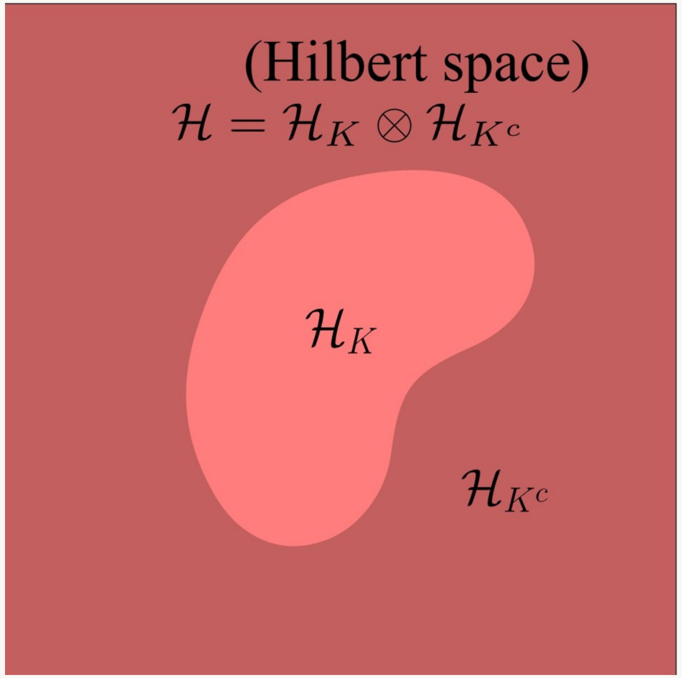
Equivalent recovery statement

There exists a quantum recovery channel acting on K^c such that:

$$\mathcal{R}(\mathrm{Tr}_K[|\xi\rangle\langle\xi|]) = |\xi\rangle\langle\xi|$$

The analogy





PENROSE TILINGS

Classical, geometric configurations

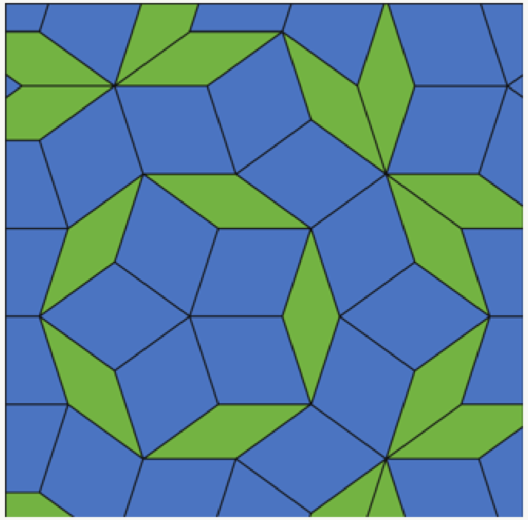
same patch appears at different locations

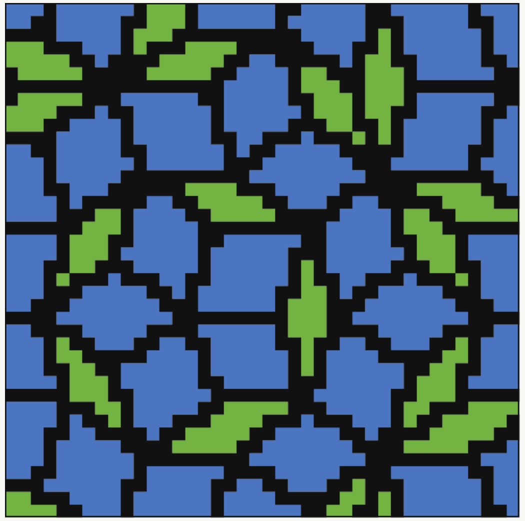
QUANTUM CODE

quantum states, including superposition;
same region K does not reveal quantum info

Can we draw a precise connection?

Tiling as a product state





Local qubit/qutrit basis

$$|g\rangle \equiv |0\rangle, |b\rangle \equiv |1\rangle, \quad (|e\rangle \equiv |2\rangle)$$

Many-body Hilbert space

$$\mathcal{H} = \bigotimes_p \mathbb{C}^3, \qquad |T\rangle = \bigotimes_p |s_p(T)\rangle$$

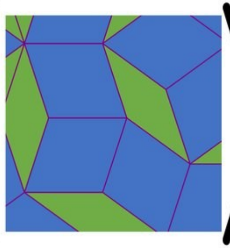
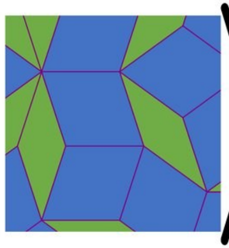
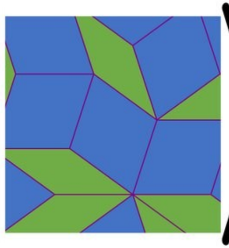
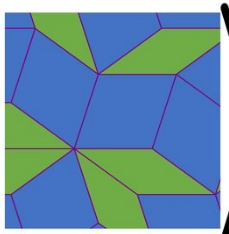
$$\langle T'|T\rangle = \delta(T', T)$$

$$|T\rangle = |T\rangle_K \otimes |T\rangle_{K^c}$$

A single $|T\rangle$ is a classical configuration—no code yet.

Codeword: Coherent superposition of tilings

$$|\Psi_{[T]}\rangle = \int dg\, |gT\rangle$$

$|\Psi_{[T]}\rangle = |$  $\rangle + |$  $\rangle + |$  $\rangle + |$  $\rangle + \cdots$

$g \in E(2) = \mathbb{R}^2 \rtimes SO(2)$ translation and/or rotation.

gT: the same tiling, translated and/or rotated

$$[T] = \{ gT : g \in E(2) \}$$

The Penrose tiling code space

$$\mathcal{C} = \operatorname{span}\{\, |\Psi_{[T]}\rangle : [T] \in \mathcal{T} \,\}, \qquad |\xi\rangle = \sum_{[T]} c_{[T]} |\Psi_{[T]}\rangle$$

Orthogonality $\langle \Psi_{[T']} | \Psi_{[T]} \rangle = 0 \text{ for } [T'] \neq [T]$

Logical data the amplitudes $c_{[T]}$ across global tiling classes

Physical data coherent superposition of the geometry

The QECC criterion in basis vectors

Let $\{\,|\psi_i\rangle\,\}$ span the code space C. Erasure of K is correctable if and only if

$$\rho_K(\xi) = \mathrm{Tr}_{K^c}(|\xi\rangle\langle\xi|) \text{ independent of } |\xi\rangle \in \mathcal{C}$$

$$Tr_{K^c}(|\psi_i\rangle\langle\psi_j|) = \langle\psi_j|\psi_i\rangle\rho_K, \qquad \forall\, i, j.$$

Diagonal terms

The reduced state on K is the same for every logical basis state:

$$Tr_{K^c}(|\psi_i\rangle\langle\psi_i|) = \rho_K \quad (i \text{ independent})$$

Off-diagonal terms

For orthogonal logical basis states:

$$Tr_{K^c}(|\psi_i\rangle\langle\psi_j|) = 0, \qquad i \neq j.$$

Off-diagonal: Recoverability

$$\mathrm{Tr}_{K^c} |\Psi_{[T]}\rangle\langle\Psi_{[T']}| = \iint dg\, dg' \; \delta\big((gT)_{K^c}, (g'T')_{K^c}\big)\, |gT\rangle_K\langle g'T'|_K$$
$$= 0 \qquad ([T] \neq [T']).$$

vanish when $[T] \neq [T']$:

1. A term survives Tr_{K^c} only if the two outside configurations agree:

$$(gT)_{K^c} = (g'T')_{K^c}$$

2. Classical recoverability implies $gT = g'T'$ globally, contradiction

Diagonal: local indistinguishability

$$\mathrm{Tr}_{K^c} |\Psi_{[T]}\rangle\langle\Psi_{[T]}| = \langle\Psi_{[T]}|\Psi_{[T]}\rangle\, \rho_{K,[T]},$$

$$\rho_{K,[T]} = \int dg\, |gT\rangle_K \langle gT|_K.$$

$\rho_{K,[T]}$ is determined by local patch frequencies:

1. Translate and rotate K through T; record the configuration induced on K.

2. The integral is the classical mixture of these configurations, weighted by their frequencies.

Local indistinguishability → $\rho_{K,[T]}$ independently of $[T]$.

Main result: the orbit states span a QECC

$$\mathrm{Tr}_{K^c}\,|\Psi_{[T]}\rangle\langle\Psi_{[T']}| = \begin{cases} 0, & [T] \neq [T'], \\ \langle\Psi_{[T]}|\Psi_{[T]}\rangle\,\rho_K, & [T] = [T']. \end{cases}$$

The erasure of every finite region K is correctable

$[T] \neq [T']$

outside match ⇒ global match

classical recoverability ⇒ cross term = 0

$[T] = [T']$

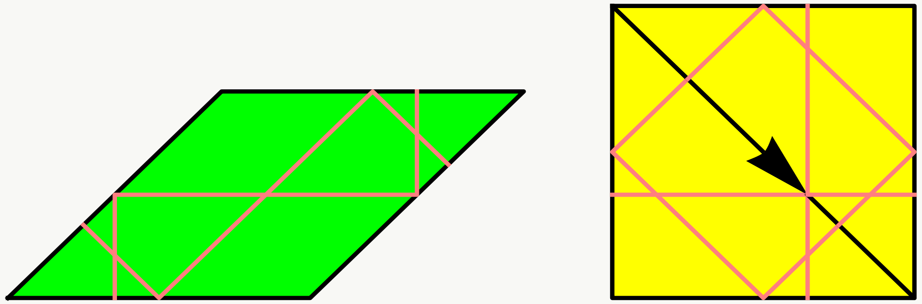
same patches + same frequencies

local indistinguishability ⇒ common ρ_K

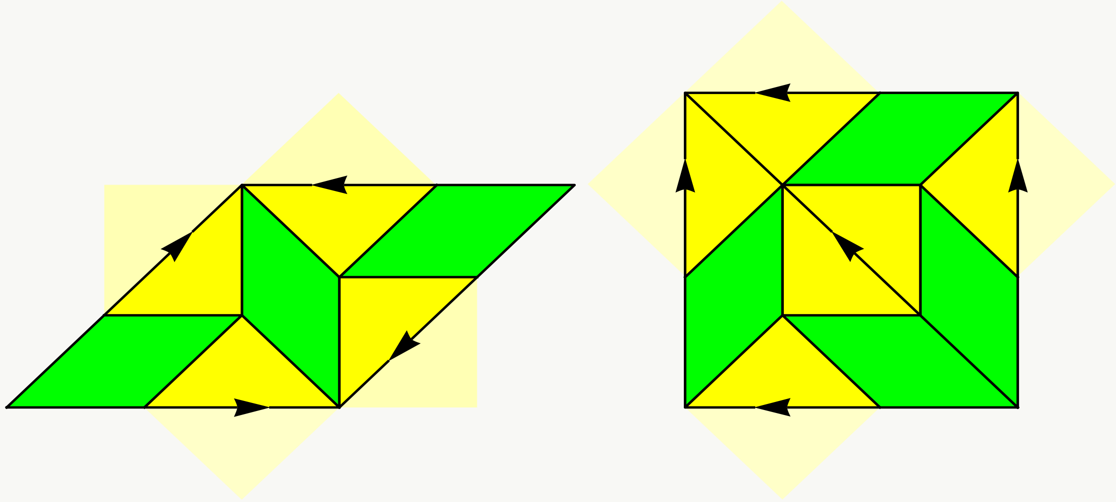
Local indistinguishability + local recoverability → quantum local indistinguishability
(⇔ quantum local recoverability)

Ammann–Beenker tilings

PROTOTILES AND CONSTRUCTION DATA



AB INFLATION RULE



Undecorated AB

USED HERE

No finite aperiodic matching rule

Genuine AB tiling

defined via AB inflation rule

Inflation / deflation forces aperiodicity

Periodic AB-like tilings

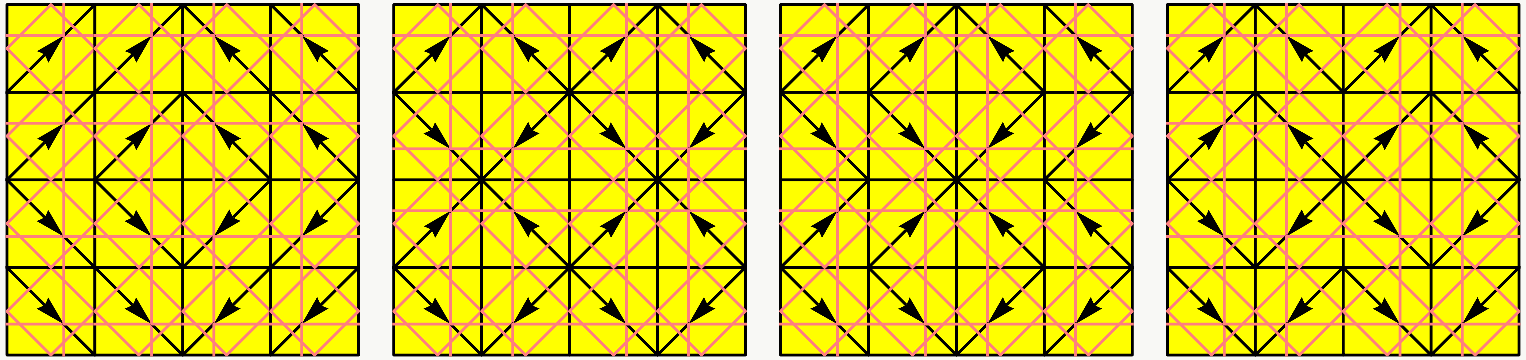
Locally looks like AB tiling

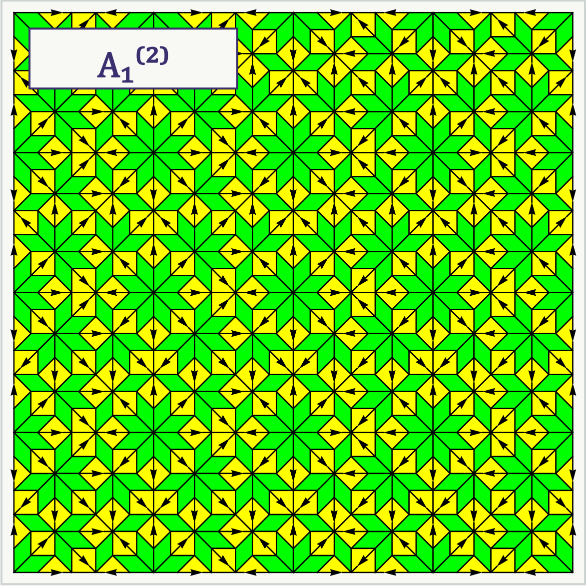
Decorated AB

Local matching rules force aperiodicity

QECC via periodic approximants

FOUR TORUS SEEDS





01 Periodic seeds

$A_i^{(0)}, i = 1, 2, 3, 4$

unbroken Ammann lines on T^2

02 inflation

$A_i^{(0)} \to A_i^{(n)}$

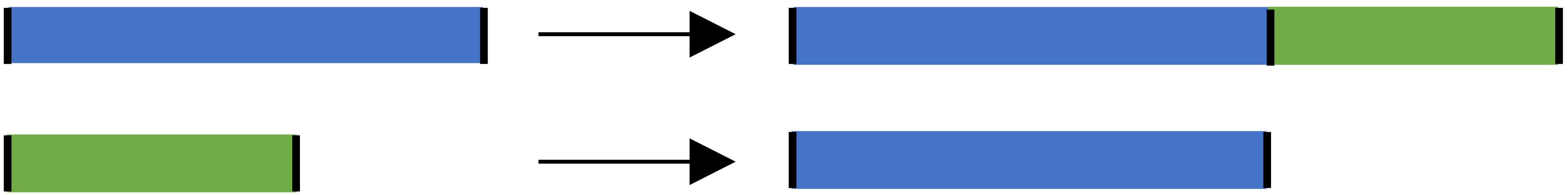
Every linear radius disk is AB-like
Locally indistinguishable

Local recovery

03 Codewords

$$|\Psi_i^{(n)}\rangle \propto \int_{\mathbb{T}^2} \mathrm{d}g\, |gA_i^{(n)}\rangle$$

Fibonacci quasicrystal: symbolic dynamics



Inflation factor $\varphi = (1 + \sqrt{5})/2$;

Substitution rule

$\mathrm{L} \leftrightarrow 1, \mathrm{S} \leftrightarrow 0, \sigma(1) = 10, \sigma(0) = 1$

Two-sided bit string

$F \in \{0,1\}^{\mathbb{Z}}$ defined by substitution rule

asymptotic ratio $N_1/N_0 \to \varphi$.

aperiodic

Fibonacci: local data and code states

Local indistinguishability

Every finite substring of F occurs in other F'.

Frequencies are independent of $[F]$

Local recoverability

A finite interval K is determined by K^c

(*except singular Fibonacci strings)

Translation-orbit wavefunction

$$|\Psi_{[F]}\rangle \propto \sum_{x \in \mathbb{Z}} |\tau_x F\rangle$$

The span of $\{ |\Psi_{[F]}\rangle\}$ is a QECC

Cyclic Fibonacci

Choose a short cyclic string $F^{(0)}$ as seed

$$F^{(\mathrm{n})} = \sigma^{\mathrm{n}}(F^{(0)}), \qquad \sigma(1) = 10, \sigma(0) = 1$$

$|F^{(\mathrm{n})}| = k_0\, f_n + k_1\, f_{n+1}$ **(k_0: number of zeros in $F^{(0)}$)**

Properties:

Length f_{n+1} substrings occur with multiplicities fixed by (k_0, k_1)

For $|K| \leq f_n + 1$, recovery from K^c is unique*

Cyclic translation superposition

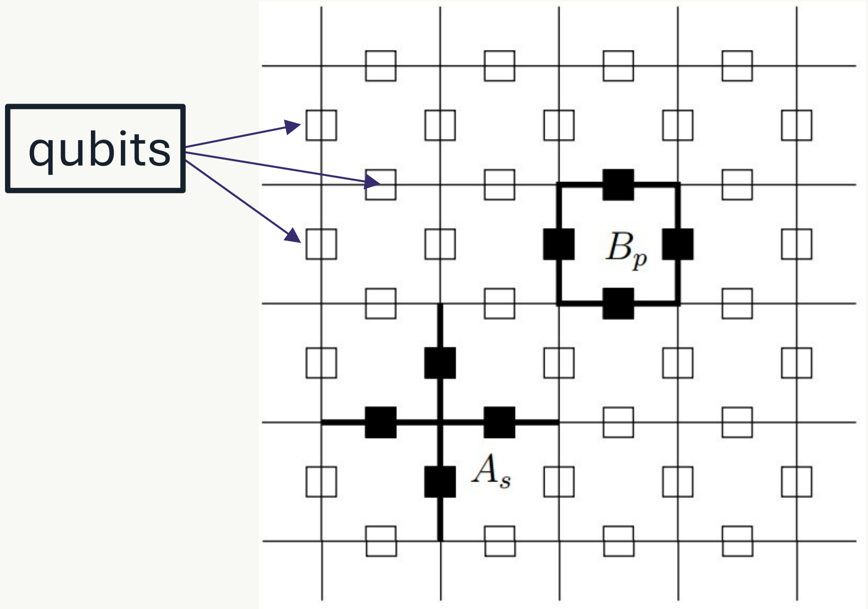
$$|\Psi_{F^{(0)}}\rangle \propto \Sigma_x \,|\, \tau_x\, F^{(\mathrm{n})}\rangle$$

A family of quasicrystal-based codes

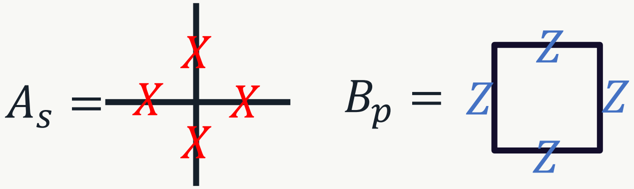
CONSTRUCTION	TYPE	GEOMETRY	CORRECTABLE REGION
Penrose orbit states	geometric	2D infinite plane	any finite connected K
Fibonacci orbit states	digital	1D infinite line	any finite interval
AB approximants	geometric	2D finite torus	disk radius $r_n \propto L$
Cyclic Fibonacci	digital	1D periodic	$\lvert K\rvert \leq f_n + 1$

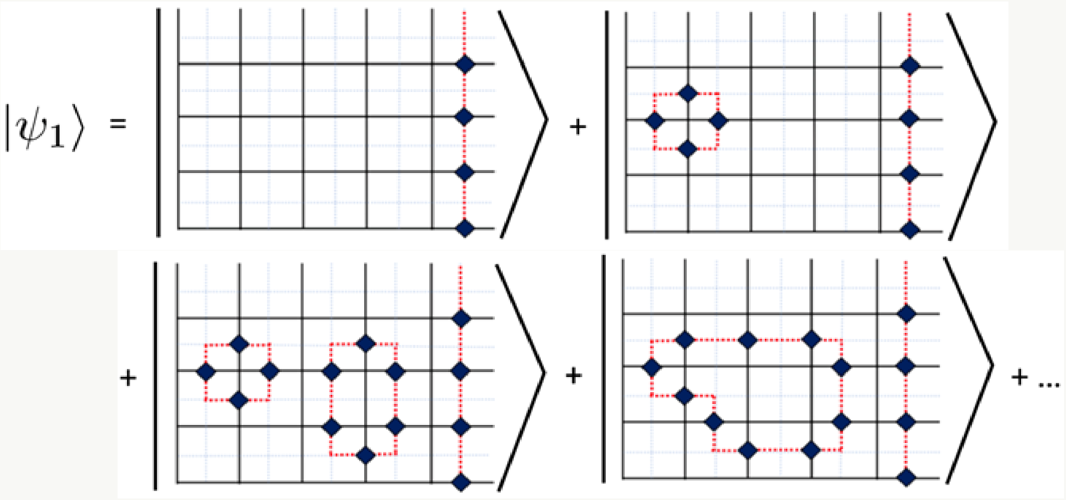
Common ingredients: orbit superposition + classical recoverability + classical local indistinguishability

Toric code: coherent closed-loop states



$$H = -\sum_s A_s - \sum_p B_p$$





$$|\psi_{ab}\rangle \propto \sum_{C \in \mathcal{L}_{ab}} |C\rangle, \qquad (a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

- Qubits live on edges.
- $\mathcal{L}$ab: configurations with winding parity (a,b).
- Four torus sectors give two logical qubits.

Logical information = global winding data.

Topology vs aperiodic geometry

TORIC CODE

$$|\psi_{ab}\rangle \propto \sum_{C \in \mathcal{L}_{ab}} |C\rangle, \qquad (a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$$

Configurations: closed loops C

Global label: winding parity / homology class (a,b)

Local moves: add or remove contractible loops

PENROSE CODE

$$|\Psi_{[T]}\rangle = \int dg\, |gT\rangle$$

Configurations: pixelized tilings gT

Global label: Euclidean orbit [T]

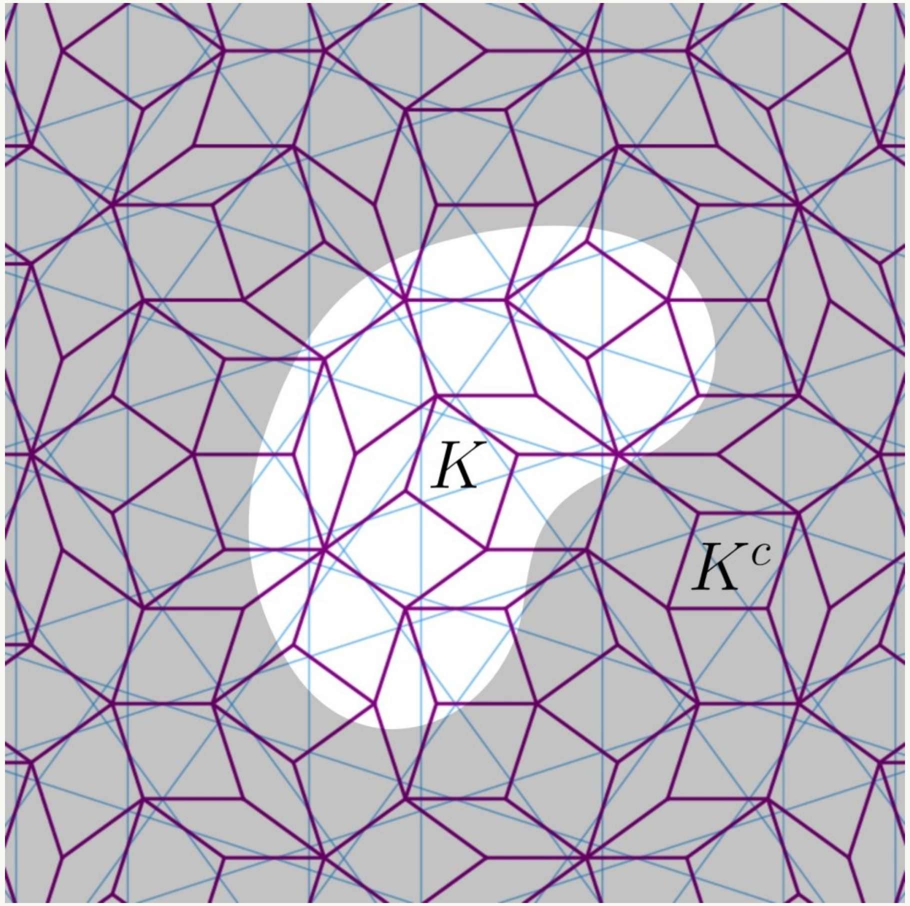
Averaged data: absolute placement and orientation

Shared structure basis code states are coherent superpositions of geometric configurations.

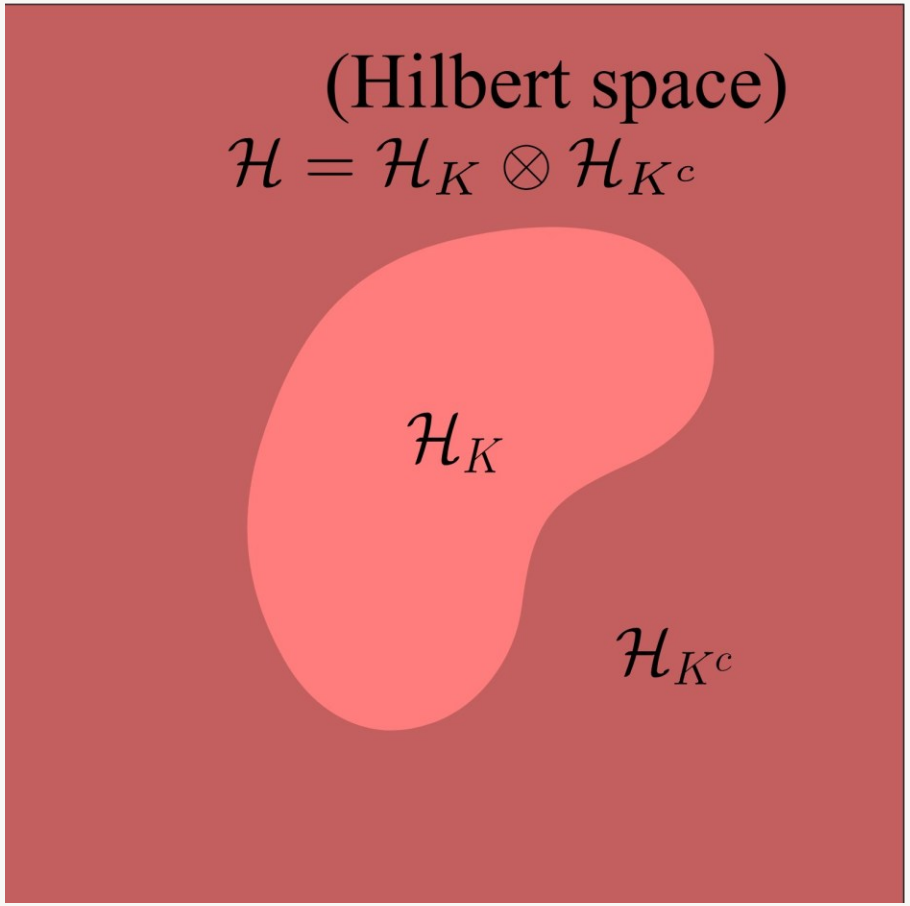
local observations cannot read the global logical label.

Difference topological winding versus globally inequivalent aperiodic geometry.

From classical geometry to quantum codes



K hides [T]; K^c reconstructs the entire tiling



K hides $|\psi\rangle$; K^c reconstructs the logical state

Quantum information can be encoded in global aperiodic geometry.